I.—ON THE METHOD OF SPHERICAL COORDINATES.

In two very ingenious papers " On the equations to loci on the surface of the sphere," published in the 12th Volume of the Edinburgh Transactions, Mr. Davies has investigated the nature and properties of curves traced on the surface of the sphere, by referring them to spherical coordinates. Although the analogy with the ordinary methods of Analytical Geometry might appear to be pretty obvious, yet, till the publication of these papers, no systematic attempt had been made to apply the method of coordinates to spherical curves; one or two isolated problems only having been solved by this means. Nor do these papers appear to have received much attention, since we do not find that the methods there developed have yet been transferred to any elementary books. We think, therefore, that it will be acceptable to our readers, to lay before them a sketch of the principles and applications of the method; not going into all the minutiæ of the subject, but still giving a sufficient number of examples to illustrate the processes. Those who are desirous of following out the subject to a farther extent, will find the original papers in the Edinburgh Transactions both interesting and instructive, and well worth their perusal. We shall not in all cases adhere closely to the methods employed by Mr. Davies, but shall change them where we think an alteration will render the treatment of the subject simpler.

As might be expected, this method bears a close analogy to Analytical Geometry of two dimensions, only substituting great circles of the sphere for straight lines. For as a straight line is the

simplest line which can be drawn on a plane, so a great circle is the simplest line which can be drawn on a spherical surface. In order, therefore, to determine the position of a point on a sphere, we assume two great circles intersecting each other at right angles for coordinate axes, the point of intersection being the origin.

Thus, in fig. 1, let $C\phi$, $C\psi$ be two great circles intersecting each other at right angles, and if P be any point on the surface of the sphere, its position with reference to C will be completely determined by drawing PM an arc of a great circle, passing through P, and perpendicular to CM; for knowing the axes CM and CP, we can find P. CM, PM are the spherical coordinates of P, and we shall generally represent them by ϕ and ψ. As in Analytical Geometry, we shall assume axes measured in the directions $C\phi$ and $C\psi$ to be positive, and those measured in the directions $C\phi'$, $C\psi'$ to be negative. This is not absolutely necessary, since a negative arc ϕ' is equal to a positive arc $2\pi - \phi'$, which may therefore be used in its place, and the results will be the same.

We may also refer spherical curves to polar coordinates, though there is not nearly so much distinction between the two methods in spherical as in rectilinear Geometry. If in fig. 1 we produce the arc MP, it will meet the axis of ψ in a point O, which is the pole of the great circle, which we have taken as the axis of ϕ. If, then, we take O as the origin, OP as the radius vector, and COP as the angle vector, the position of the point P may be determined by means of a relation between these two quantities. It will be seen at once from the figure, that the angle vector is equal to ϕ in the other method, and that the radius vector is the complement of ψ. There is, therefore, very little difference in expressions for curves by either method, and the one can be easily transferred to the other. Mr. Davies has generally made use of polar coordinates: in the following pages we shall chiefly employ the other.

We shall now proceed to the investigation of equations to lines traced on the sphere, beginning with a great circle.

1. To find the equation to a great circle. Let PAB (fig. 1.) be a great circle cutting the axes in A and B. Let $CM = \phi$, $PM = \psi$, $CA = a$, $CB = \beta$, $PA\phi = \iota$.

Then, by Napier's rules,
$$\sin AM = \tan PM \cdot \cot PAM;$$
therefore $\tan \psi = - \tan \iota \sin (a - \phi)$:
$$\text{when } \phi = 0, \quad \psi = \beta;$$
therefore $\tan \beta = - \tan \iota \sin a$:
whence $\tan \psi = \dfrac{\tan \beta \sin (a - \phi)}{\sin a} = - \dfrac{\tan \beta}{\sin a} \sin (\phi - a).$

The general equation to a great circle may therefore be put under the form
$$\tan \psi = m \sin (\phi - a)\ldots\ldots\ldots(1),$$

$$\text{where } m = - \frac{\tan \beta}{\sin \alpha}.$$

If it pass through a given point ϕ_1, ψ_1, we have

$$\tan \psi_1 = m \sin (\phi_1 - \alpha);$$

whence

$$\tan \psi = \frac{\tan \psi_1}{\sin (\phi_1 - \alpha)} \sin (\phi - \alpha)$$

is the equation to a great circle passing through a given point ϕ_1, ψ_1.

If it pass through two given points, the same method may be used, but the following is more convenient.

The general equation

$$\tan \psi = m \sin (\phi - \alpha)$$

may be put under the form

$$\tan \psi = A \sin \phi + B \cos \phi,$$

or the still more general one

$$\tan \psi = A \sin (\phi - \alpha) + B \sin (\phi - \beta);$$

whence we have

$$\tan \psi_1 = A \sin (\phi_1 - \alpha) + B \sin (\phi_1 - \beta),$$
$$\tan \psi_2 = A \sin (\phi_2 - \alpha) + B \sin (\phi_2 - \beta),$$

ψ_1, ϕ_1, ψ_2, ϕ_2 being the coordinates of the two given points.

As we have four indeterminate constants and only two equations, we may assume $\alpha = \phi_1$, $\beta = \phi_2$;

$$\text{therefore } \tan \psi_1 = - B \sin (\phi_2 \quad \phi_1),$$
$$\text{and } \tan \psi_2 = A \sin (\phi_2 - \phi_1);$$

$$\text{therefore } A = \frac{\tan \psi_2}{\sin (\phi_2 - \phi_1)}, \text{ and } B = \frac{\tan \psi_1}{\sin (\phi_2 - \phi_1)}.$$

Hence the equation to a great circle passing through two given points ϕ_1, ψ_1, ϕ_2, ψ_2, is

$$\tan \psi \sin (\phi_2 - \phi_1) = \tan \psi_2 \sin (\phi - \phi_1) - \tan \psi_1 \sin (\phi \quad \phi_2).$$

2. To find the coordinates of the intersection of two arcs.

Let the equation be

$$\tan \psi = m_1 \sin (\phi - \alpha_1) = m_1 (\sin \phi \cos \alpha_1 - \cos \phi \sin \alpha_1),$$
$$\tan \psi = m_2 \sin (\phi - \alpha_2) = m_2 (\sin \phi \cos \alpha_2 - \cos \phi \sin \alpha_2).$$

When the coordinates are common, we have, by dividing one by the other,

$$1 = \frac{m_1}{m_2} \frac{\tan \phi \cos \alpha_1 - \sin \alpha_1}{\tan \phi \cos \alpha_2 - \sin \alpha_2},$$

whence

$$\tan \phi = \frac{m_1 \sin \alpha_1 - m_2 \sin \alpha_2}{m_1 \cos \alpha_1 - m_2 \cos \alpha_2};$$

therefore

$$\sin \phi = \frac{m_1 \sin \alpha_1 - m_2 \sin \alpha_2}{\{m_1^2 - 2m_1 m_2 \cos (\alpha_1 - \alpha_2) + m_2^2\}^{\frac{1}{2}}},$$

and $\cos \phi = \dfrac{m_1 \cos a_1 - m_2 \cos a_2}{\{m_1{}^2 - 2m_1 m_2 \cos (a_1 - a_2) + m_2{}^2\}^{\frac{1}{2}}}.$

Substituting then in the expression for $\tan \psi$, we find, after reduction,

$$\tan \psi = \frac{m_1 m_2 \sin (a_1 - a_2)}{\{m_1{}^2 - 2m_1 m_2 \cos (a_1 - a_2) + m_2{}^2\}^{\frac{1}{2}}}.$$

3. To find the angle between two great circles whose equations are given.

Let AQ, A′Q be the circles, and let their equations be

$$\tan \psi = m_1 \sin (\phi - a_1),$$
$$\tan \psi' = m_2 \sin (\phi - a_2).$$

Now, in the triangle AQA′,

$\cos AQA = \cos AA' \sin QAA' \sin QA'A - \cos QAA \cos QA'A,$

and $AA' = a_1 - a_2, \ \tan QAA' = m_1, \ \tan QA'A = -m_2;$

whence $\cos AQA' = \dfrac{m_1 m_2 \cos (a_1 - a_2) + 1}{\sqrt{1 + m_1{}^2} \sqrt{1 + m_2{}^2}}.$

If the circles cut each other at right angles,

$$\cos AQA' = 0,$$

and the condition is

$$m_1 m_2 \cos (a_1 - a_2) + 1 = 0.$$

4. To find the length of the arc joining two points in terms of the coordinates of the points.

Let PQ (fig. 3) be the points whose coordinates are $\phi_1, \psi_1, \phi_2, \psi_2$. Produce MP, NQ to meet in A the pole of CMN. Then

$$\cos PQ = \sin AP \sin AQ \cos PAQ + \cos AP \cos AQ,$$

or $\cos PQ = \cos \psi_1 \cos \psi_2 \cos (\phi_2 - \phi_1) + \sin \psi_1 \sin \psi_2.$

Hence the equation to a small circle, whose radius is the radius of the sphere multiplied by $\sin \gamma$, or whose distance from its pole is γ, and the coordinates of whose centre are ϕ_1, ψ_1, is

$$\cos \psi_1 \cos \psi \cos (\phi - \phi_1) + \sin \psi_1 \sin \psi = \cos \gamma.$$

5. To find the equation to the perpendicular from a given point on a given great circle.

Let the equation to the given great circle be

$$\tan \psi = m \sin (\phi - a),$$

ϕ_1, ψ_1 the coordinates of the given point. Then, if the required equation be of the form

$$\tan \psi = m_1 \sin (\phi - a_1),$$

we find $m_1 = \dfrac{\tan \psi_1}{\sin (\phi_i - a_1)},$

and $\dfrac{m \tan \psi_1}{\sin (\phi_1 - a_1)} \cos (a_1 - a) + 1 = 0,$

as a condition for determining a_1 by (3). Whence

$$m \tan \psi_1 (\cos a_1 \cos a + \sin a_1 \sin a)$$
$$= - (\sin \phi_1 \cos a_1 - \cos \phi_1 \sin a_1),$$

which gives

$$\tan a_1 = \frac{\sin \phi_1 + m \tan \psi_1 \cos a}{\cos \phi_1 - m \tan \psi_1 \sin a}.$$

Therefore

$$\tan \psi = \frac{\tan \psi_1 \sin (\phi - a_1)}{\sin (\phi_1 - a_1)} = \tan \psi_1 . \frac{\sin \phi - \tan a_1 \cos \phi}{\sin \phi_1 - \tan a_1 \cos \phi_1}$$

$$= \tan \psi_1 . [(\cos \phi_1 - m \tan \psi_1 \sin a) \sin \phi$$
$$- (\sin \phi_1 + m \tan \psi_1 \cos a) \cos \phi]$$

divided by

$$[(\cos \phi_1 - m \tan \psi_1 \sin a) \sin \phi_1 - (\sin \phi_1 + m \tan \psi_1 \cos a) \cos \phi_1];$$

whence, after reduction,

$$\tan \psi = - \frac{1}{m \cos (\phi_1 - a)} \{\sin (\phi - \phi_1) - m \tan \psi_1 \cos (\phi - a)\}.$$

6. To find the length of the perpendicular from a given point on a given great circle.

Let AR (fig. 4) be the circle whose equation is

$$\tan \psi = m \sin (\phi - a),$$

Q the given point whose coordinates are ϕ_1, ψ_1.

Then
$$\tan PN = m \sin (\phi_1 - a),$$
also $\sin QR = \sin PQ . \sin QPR,$
but $\sin QPR = \sin APN,$
and $\cos PAN = \sin APN \cos PN,$

so that $\sin QPR = \dfrac{\cos PAN}{\cos PN};$

therefore $\sin QR = \dfrac{\sin PQ \cos PAN}{\cos PN}$

$$= \frac{\sin (QN - PN) \cos PAN}{\cos PN}$$

$$= \sin QN - \cos QN \tan PN \cos PAN$$

$$= \frac{\sin \psi_1 - m \cos \psi_1 \sin (\phi - a)}{\sqrt{1 + m^2}}.$$

7. The equations we have arrived at serve to demonstrate readily that the perpendiculars on the sides of a spherical triangle from the opposite angles meet in one point.

Let AB, Cc (fig. 5) be the axes, cA $= a$, cB $= \beta$. The equation to AC, since it cuts the axis at a distance $- a$, and is inclined to it at an angle A, is

$$\tan \psi = \tan A \sin (\phi + a).$$

The equation to BC which cuts the axis at a distance β, and is inclined to it at an angle $\pi - \beta$, is

$$\tan \psi = - \tan \beta \sin (\phi - \beta).$$

The equation to Bb is by (3) making $\phi_1 = \beta$, $\psi_1 = 0$,

$$\tan \psi = - \frac{1}{\tan A \cos (\alpha + \beta)} \sin (\phi - \beta).$$

Similarly the equation to Aa is

$$\tan \psi = \frac{1}{\tan B \cos (\alpha + \beta)} \sin (\phi + \alpha).$$

Put $\phi = 0$ in these equations, and we have

$$\tan \psi_1 = \frac{\sin \beta}{\tan A \cos (\alpha + \beta)}$$

$$\tan \psi_2 = \frac{\sin \alpha}{\tan B \cos (\alpha + \beta)}.$$

But $\quad \sin \alpha = \cot A \tan Cc, \quad \sin \beta = \cot B \tan Cc.$

Therefore the values of $\tan \psi_1$, $\tan \psi_2$ are identical, each being equal to $\dfrac{\cot A \cot B \tan Cc}{\cos AB}$.

8. We shall now proceed to the consideration of some more complicated spherical curves, beginning with one which, being defined in the same manner as a plane ellipse, is called a spherical ellipse.

To find the locus of a point, the sum of whose distances from two given points is constant.

Let S, H (fig. 6) be the two fixed points; take the arc joining them for the axis of ϕ, and the middle point of it for the origin.

Let $\quad$ SH $= 2\gamma$, SP $+$ HP $= 2a$, CM $= \phi$. PM $= \psi$.

Then $\qquad \cos$ SP $= \cos (\gamma - \phi) \cos \psi$

$$\cos \text{HP} = \cos (\gamma + \phi) \cos \psi.$$

Adding and reducing,

$$\cos \tfrac{1}{2} (\text{HP} - \text{SP}) = \frac{\cos \gamma \cos \phi \cos \psi}{\cos a}.$$

Similarly, by subtraction,

$$\sin \tfrac{1}{2} (\text{HP} - \text{SP}) = \frac{\sin \gamma \sin \phi \cos \psi}{\sin a}.$$

Squaring and adding,

$$\left\{ \left(\frac{\cos \gamma \cos \phi}{\cos a} \right)^2 + \left(\frac{\sin \gamma \sin \phi}{\sin a} \right)^2 \right\} (\cos \psi)^2 = 1,$$

or $(\sec \psi)^2 = \left(\dfrac{\cos \gamma}{\cos a} \right)^2 - \dfrac{(\cos \gamma \sin a)^2 - (\sin \gamma \cos a)^2}{(\cos a \sin a)^2} (\sin \phi)^2.$

Therefore

$$(\tan \psi)^2 = \frac{(\cos \gamma)^2 - (\cos a)^2}{(\cos a)^2} - \frac{(\sin a)^2 - (\sin \gamma)^2}{(\cos a \sin a)^2} (\sin \phi)^2$$

$$= \frac{(\sin a)^2 - (\sin \gamma)^2}{(\cos a \sin a)^2} \{(\sin a)^2 - (\sin \phi)^2\},$$

when $\phi = 0$ $(\tan \psi)^2 = \dfrac{(\sin a)^2 - (\sin \gamma)^2}{(\cos a)^2} = (\tan \beta)^2$ suppose.

Then $\quad (\tan \psi)^2 = \left(\dfrac{\tan \beta}{\sin a}\right)^2 \{(\sin a)^2 - (\sin \phi)^2\},$

$$\text{or} \left(\frac{\tan \psi}{\tan \beta}\right)^2 + \left(\frac{\sin \phi}{\sin a}\right)^2 = 1,$$

which is the final equation.

In the spherical, as in the plane ellipse, the distance between the focus and the extremity of the axis minor is equal to the semi-axis major, for we have

$$(\tan \beta)^2 = \frac{(\sin a)^2 - (\sin \gamma)^2}{(\cos a)^2};$$

whence $\sec \beta = \dfrac{\cos \gamma}{\cos a}$,

and $\cos a = \cos \beta \cos \gamma.$

But $\qquad \cos \beta \cos \gamma = \cos SB,$

and therefore $SB = a = CA.$

The spherical ellipse and hyperbola are the same. For if the sum of the distances of P from S and H be constant, the difference of the distances of P from S and a point opposite to H is also constant, and therefore the locus of P is also the locus of a curve traced according to the definition of a hyperbola.

9. The locus of the vertex of a right-angled spherical triangle whose base is given is a spherical ellipse.

Take the middle point of the base as origin, the axis of ϕ coinciding with the base, and let the length of the base be $2a$. The equations to the two sides will be

$$\tan \psi = m_1 \sin (\phi - a)$$
$$\tan \psi = m_2 \sin (\phi + a),$$

and since they are at right angles to each other,

$$m_1 m_2 \cos 2a + 1 = 0 \text{ by (3.)}$$

Therefore $\quad (\tan \psi)^2 = - \dfrac{\sin (\phi + a) \sin (\phi - a)}{\cos 2a},$

$$\text{or} \ (\tan \psi)^2 = \frac{(\sin a)^2 - (\sin \phi)^2}{\cos 2a},$$

which is the equation to an ellipse.

If β be the value of ψ when $\phi = 0$,

$$(\tan \beta)^2 = \frac{(\sin a)^2}{\cos 2a},$$

whence $\sin \beta = \tan a$.

10. The curve of intersection of an ellipsoid with a sphere is a spherical ellipse.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Put $x = r \cos \phi \cos \psi$, $y = r \sin \phi \cos \psi$, $z = r \sin \psi$.

Then

$$(\cos \psi)^2 \left\{ \frac{(\sin \phi)^2}{b^2} + \frac{(\cos \phi)^2}{a^2} \right\} + \frac{(\sin \psi)^2}{c^2} = \frac{1}{r^2}.$$

Dividing by $(\cos \psi)^2$, and putting $1 - (\sin \phi)^2$ for $(\cos \phi)^2$,

$$\frac{(\tan \psi)^2}{c^2} + (\sin \phi)^2 \left\{ \frac{1}{b^2} - \frac{1}{a^2} \right\} + \frac{1}{a^2} = \frac{1}{r^2} + \frac{(\tan \psi)^2}{r^2}.$$

Therefore

$$(\tan \psi)^2 \left(\frac{1}{c^2} - \frac{1}{r^2} \right) + (\sin \phi)^2 \left(\frac{1}{b^2} - \frac{1}{a^2} \right) = \frac{1}{r^2} - \frac{1}{a^2}.$$

Since, in order that the surfaces may intersect, we must have $r < a$ and $> c$; this is the equation to a spherical ellipse whose semi-axes are

$$\sin^{-1} \left\{ \frac{\frac{1}{r^2} - \frac{1}{a^2}}{\frac{1}{b^2} - \frac{1}{a^2}} \right\}^{\frac{1}{2}} \text{ and } \tan^{-1} \left\{ \frac{\frac{1}{r^2} - \frac{1}{a^2}}{\frac{1}{c^2} - \frac{1}{r^2}} \right\}^{\frac{1}{2}}.$$

The section will be a great circle when the major axis becomes equal to π, or $\frac{1}{r^2} - \frac{1}{a^2} = \frac{1}{b^2} - \frac{1}{a^2}$, or $r = b$, in which case the equation becomes

$$\tan \psi = \left\{ \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{c^2} - \frac{1}{b^2}} \right\}^{\frac{1}{2}} \cos \phi;$$

therefore the tangent of the inclination of this circle to that of ϕ is

$$\left\{ \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{c^2} - \frac{1}{b^2}} \right\}^{\frac{1}{2}}.$$

11. The next curve we shall consider is the equable spherical spiral, of which a particular case is the spiral of Pappus. Its definition is: If a meridian PRP' (fig. 7) revolve uniformly about

an axis PP' of a sphere, while a point M moves from P uniformly along PRP' from P to P', the locus of the point M is the equable spherical spiral. Take P as the origin of the polar coordinates, EPM as the angle vector $= \theta$, PM the radius vector $= \phi$, and let the ratio of the motions be $m : n$; then the required equation is

$$m\phi = n\theta.$$

To discuss this equation, we shall assume different relations between m and n.

1. Let $m = n$; then $\phi = \theta$.

During the first quadrant the point M_1 (fig. 8) will be in the spherical octant PEO, and at the end of the first quadrant it will be at O.

During the second quadrant it will be in the octant OP'Q, and at π of longitude it will be at P'.

During the third quadrant the radius vector will be measured on a meridian $PR_1P'M_3$, of which PR_1P' is on the posterior surface of the sphere; but as ϕ is also greater than π, the point M_3 will be on the convex side of the sphere in the octant EOP'; and at $\frac{3\pi}{2}$, M_3 will be at O.

During the fourth quadrant it will be in the octant POQ, and at the end will return to P, and after that it will retrace the same path.

When $m = 2n$, or $\phi = \frac{1}{2}\theta$, (fig. 9) is the locus, where ON, ON', are equal to $\frac{\pi}{4}$.

When $m = 4n$, or $\phi = \frac{1}{4}\theta$, we have the particular case considered by Pappus. Here ϕ does not acquire the value $\frac{\pi}{2}$, or the curve does not cut the equator till after a complete revolution of the meridian. After two revolutions it is at the opposite pole, and after four revolutions it returns to the origin.

Generally, if m and n be commensurable, the branches of the spiral will return in the same order, and coalesce when $\theta = 2mn\pi$, but if they be incommensurable this will never occur.

12. If a cylinder, whose radius is half the radius of a sphere, and the centre of whose base is placed at the distance of half the radius from the centre, intersect the sphere, the curve of intersection is that equable spiral whose equation is

$$\phi = \theta.$$

The equation to the cylinder is (the radius of the sphere being 1)

$$y^2 + (x - \tfrac{1}{2})^2 = \tfrac{1}{4};$$

where it meets the sphere,

$$y = \cos\theta \sin\phi, \quad x = \cos\theta \cos\phi;$$

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therefore $(\cos \theta)^2 - \cos \theta \cos \phi + \frac{1}{4} = \frac{1}{4}$:

whence $\cos \theta = \cos \phi$,

and $\qquad \theta = \pm \phi$.

13. To find the path of the vertical projection of the Sun, supposing the Sun to move round the Earth in a circle with an equable motion.

Let EQ (fig. 10) be the equator, SQ the ecliptic, P, M their respective poles. Then, in the right-angled triangle PSR, considering the angle at S as the right angle,

$$\cos PR = \cos SP \cdot \cos SR.$$

Let $PR = \phi$, $PM =$ obliquity of ecliptic $= \omega$;

then $\cos \phi = - \sin \omega \cos SR$.

Now, if n be the ratio of the Earth's angular velocity about its axis to that about the Sun, $\theta = n SR$; therefore

$$\cos \phi = - \sin \omega \cos \frac{\theta}{n}$$

is the equation to the path.

If $\omega = \dfrac{\pi}{2}$, this becomes the equable spherical spiral.

14. The last curve which we shall consider, is the Rhumb line or Loxodrome, which has always attracted much attention, from its use in navigation. The definition of this curve is, that it cuts all the meridians at equal angles.

Let P (fig. 11) be the pole, PE, PQ two successive meridians, making an angle $d\theta$ with each other, θ being the longitude of PM measured from a given point. Let $PM = \phi$. Draw MR parallel to the equator; then $RN = d\phi$: hence $MR = \sin \phi d\phi$. Then, considering the ultimate elements of the arcs which form the elemental triangle MNR, as straight lines, we have

$$\tan MNR = \frac{MR}{RN}.$$

As the angle MNR is to be the same for every meridian, let it $= \alpha$. Then

$$\frac{\sin \phi d\theta}{d\phi} = \tan \alpha,$$

$$\text{or } \frac{d\phi}{\sin \phi} = \cot \alpha d\theta.$$

Whence integrating

$$\log \tan \frac{\phi}{2} = \theta \cot \alpha + C.$$

To determine C. Let $\phi = \dfrac{\pi}{2}$ when $\theta = 0$. Then $C = 0$.

The equation may therefore be put under the form

$$\tan \frac{\phi}{2} = \varepsilon^{\theta \cot \alpha}.$$

15. It is frequently an interesting problem to find what are the plane curves produced by the projections of spherical curves. For this purpose we shall investigate a general formula for such transformation.

Let A (fig. 12) be the pole to which the curves have been referred, C the centre of the circle, P a point in the curve, O any point situate in the axis AB, from which the point P is to be projected on a plane MR parallel to the equator.

Let $AC = r$, $CO = a$, $CM = b$, $AP = \phi$, $MR = \rho$.

It is clear that the projection of the angle between two meridians will be equal to the original angle, since the projecting planes will pass through AO. Hence if we consider the curve on MR as referred to polar coordinates, the angle vector will remain the same, and we have only to determine the radius vector $MR = \rho$ in terms of ψ.

Now from the similar triangles RMO, PNO,

$$RM : OM = PN : ON,$$
$$\rho : a + b = r \sin \phi : a + r \cos \phi.$$

Whence
$$\rho = \frac{(a + b)\, r \sin \phi}{a + r \cos \phi}.$$

We shall confine our attention to the three principal projections: the orthographic, the stereographic, and the gnomonic.

In the orthographic the projecting lines are parallel, therefore a is infinite, and we find

$$\rho = r \sin \psi.$$

In the stereographic $a = r$ and $b = c$, therefore

$$\rho = \frac{r \sin \phi}{1 + \cos \phi} = r \tan \frac{\phi}{2}.$$

In the gnomonic $a = 0$, $b = r$, therefore

$$\rho = r \tan \phi.$$

Applying these expressions to the equation to the rhumb line, we have for its orthographic projection

$$\rho = \frac{2r}{\varepsilon^{\theta \cot \alpha} + \varepsilon^{-\theta \cot \alpha}}.$$

For the stereographic

$$\rho = r\varepsilon^{\theta \cot \alpha}.$$

For the gnomonic

$$\rho = \frac{2r}{\varepsilon^{-\theta \cot \alpha} - \varepsilon^{\theta \cot \alpha}},$$

which are three of Cotes's spirals.

There are many other interesting points regarding spherical curves, such as their curvature, rectification, and quadrature; but we shall defer the consideration of these till a future Number.

$\int\!\int .$

II.—ON SOME PROPERTIES OF THE PARABOLA.*

There are many very interesting properties of the Conic Sections which are not to be found in the usual works on the subject, but are scattered through various memoirs in scientific Journals. Those relating to the properties of polygons inscribed in and circumscribed round conic sections, have been investigated by a great many writers both in France and England. Pascal was the first who engaged in these researches, and was led by the curious properties which he discovered to call one of these polygons the "hexagramme mystique." After him Maclaurin gave a proof of a theorem which is not only beautiful in itself, but also very fertile in its consequences. In more recent times Biranchon has demonstrated the remarkable theorems, that in all hexagons either inscribed in or circumscribed round a conic section, the three diagonals joining opposite angles will intersect in one point. Subsequently, Davies in this country, and Dandelin in Belgium, proved in different ways the same propositions along with others. The latter adopted a very peculiar method, deducing these and many other properties of sections of the cone by considering the cone as a particular case of the "hyperboloide gauche." Generally speaking the Geometrical method is more easily applied than the Analytical to these cases, and accordingly all the proofs given have depended on geometry, with the exception of one published by Mr. Lubbock in the Number of the Philosophical Magazine for August 1838. He has there demonstrated, by analysis, Brianchon's Theorem for a circumscribing hexagon in the particular case where the conic section is a parabola; but his method is tedious, and not remarkable for symmetry and elegance, so that another proof is still desirable. The following one is founded on the form of the equation to the tangent of the parabola which is given in Art. 2 of our first Number.

Let the parabola be referred to its vertex, then the equation to its tangent by that article is

$$y = \frac{x}{a} + ma,$$

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* From a Correspondent.

where a is the tangent of the angle which the tangent makes with the axis of y. If a' be the corresponding quantity for another tangent, its equation will be

$$y = \frac{x}{a'} + ma'.$$

Combining these equations, we shall find for the coordinates of the point of intersection of the two tangents

$$x = maa', \quad y = m(a + a').$$

We shall distinguish the tangents which form the different sides of the hexagon by suffixing numbers to the a which determines their position, and we shall likewise distinguish the coordinates of the summits of the hexagon by suffix letters.

The equations to the three diagonals are these:

(1) $y(a_4 a_5 - a_1 a_2) - x(a_4 + a_5 - a_1 - a_2) =$
$$m\{(a_1 + a_2)a_4 a_5 - (a_4 + a_5)a_1 a_2\}.$$

(2) $y(a_5 a_6 - a_2 a_3) - x(a_5 + a_6 - a_2 - a_3) =$
$$m\{(a_2 + a_3)a_5 a_6 - (a_5 + a_6)a_2 a_3\}.$$

(3) $y(a_6 a_1 - a_3 a_4) - x(a_6 + a_1 - a_3 - a_4) =$
$$m\{(a_3 + a_4)a_1 a_6 - (a_1 + a_6)a_3 a_4\}.$$

Expressions which, as they ought to be, are symmetrical with respect to the a's.

Multiply (1) by a_6, (2) by $-a_4$, (3) by a_2, and add. Then y will disappear, and we shall find

$$x = m\,\frac{a_3 a_6(a_4 a_5 - a_1 a_2) - a_4 a_1(a_5 a_6 - a_2 a_3) + a_5 a_2(a_6 a_1 - a_3 a_4)}{a_1 a_2 - a_2 a_3 + a_3 a_4 - a_4 a_5 + a_5 a_6 - a_6 a_1}.$$

Again, multiply (1) by a_3, (2) by $-a_1$, (3) by a_5, and add: as before, y will disappear, and we shall find the same value for x. Consequently two straight lines whose equations are

$$(1)\ a_6 - (2)\ a_4 = 0,$$
$$\text{and } (1)\ a_3 - (2)\ a_1 = 0,$$

and which have a point in common, cut (3) in points whose abscissæ are equal, and which therefore coincide. Hence either two straight lines enclose a space, or (3) passes through the intersection of (1) and (2). Thus the existence of the point common to the three diagonals has been proved, and its abscissa found. To determine its ordinate, add (1), (2), (3), when x disappears, and we have

$$y = m\{a_1 a_2(a_4 + a_5) - a_2 a_3(a_5 + a_6) + a_3 a_4(a_6 + a_1) - a_4 a_5(a_1 + a_2)$$
$$+ a_5 a_6(a_2 + a_3) + a_6 a_1(a_3 + a_4)\}$$

divided by

$$a_1 a_2 - a_2 a_3 + a_3 a_4 - a_4 a_5 + a_5 a_6 - a_6 a_1.$$

If we call the coordinates of the point where the third and sixth

sides of the hexagon meet $x_{,,,}$ $y_{,,,}$, and so of the other two points, these expressions for x and y become

$$x = \frac{x_{,,,}(x_4 - x_1) - x_{IV}(x_5 - x_2) + x_V(x_6 - x_3)}{x_1 - x_2 + x_3 - x_4 + x_5 - x_6}$$

$$y = \frac{x_1 y_4 - x_2 y_5 + x_3 y_6 - x_4 y_1 + x_5 y_2 - x_6 y_3}{x_1 - x_2 + x_3 - x_4 + x_5 - x_6}.$$

These expressions, as of course we should expect, are symmetrical.

In the last Number of this Journal a demonstration was given of a property of a parabola: That the circle which passes through the intersections of three tangents also passes through the focus. Although six demonstrations of this theorem have already appeared, yet the following is so simple that its insertion here may not be inappropriate.

Referring the parabola to the focus as origin, we can put the equation to the tangent under the form

$$y - \frac{x}{m} = a \left(m + \frac{1}{m} \right),$$

where a is one-fourth of the parameter, and m the trigonometrical tangent of the angle which the tangent makes with the axis of y. Hence if x_1, y_1 be the coordinates of the point of intersection of

$$y - \frac{x}{m} = a \left(m + \frac{1}{m} \right),$$

with
$$y - \frac{x}{m'} = a \left(m' + \frac{1}{m'} \right),$$

we have
$$x_1 = a \, (mm' - 1),$$
$$y_1 = a \, (m + m'),$$

or putting
$$m = \frac{\sin a}{\cos a}, \quad m' = \frac{\sin a'}{\cos a'}$$

$$x_1 = a \, \frac{\cos (a + a')}{\cos a \cos a'}$$

$$y_1 = a \, \frac{\sin (a + a')}{\cos a \cos a'}.$$

To simplify these expressions turn the axes through an angle $= - (a + a' + a'')$, and if x'', y'' be the new values of the coordinates, we find, after some simple reductions,

$$x'' = \frac{a \cos a''}{\cos a \cos a'}, \quad y'' = - \frac{a \sin a''}{\cos a \cos a'}.$$

Squaring these and adding,

$$x''^2 + y''^2 = \frac{a^2}{\cos^2 a \cos^2 a'} = \frac{a}{\cos a \cos a' \cos a''} \cdot \frac{a \cos a''}{\cos a \cos a'},$$

or
$$x''^2 + y''^2 = \frac{a x''}{\cos a \cos a' \cos a''}.$$

Now this being symmetrical between a, a', a'', will hold equally true of the three points of intersection, and it is the equation to a circle passing through the origin which is the focus, whose diameter coincides with the axis of x, and whose radius is

$$\frac{a}{2 \cos a \cos a' \cos a''}.$$

The chief advantage of this method besides its simplicity is, that it gives us very readily the radius of the circle, and the position of the diameter which passes through the focus.

It is easily seen that the distances from the focus of the three points of intersection of the tangents are respectively

$$r'' = \frac{a}{\cos a \cos a'}, \quad r' = \frac{a}{\cos a \cos a''}, \quad r = \frac{a}{\cos a' \cos a''},$$

The area of the triangle formed by the intersection of the tangents, can be expressed by an elegant symmetrical function of $\tan a$, $\tan a'$, $\tan a''$, that is, of m, m', m''. Since the lines joining the origin with the vertices of the triangle make angles a, a', a'' with the diameter of the circle or the axis of x, the angles they make with each other are $a' - a$, $a'' - a'$, $a'' - a$, and the area of the triangle will be

$$\tfrac{1}{2} rr' \sin (a' - a) + \tfrac{1}{2} r'r'' \sin (a'' - a') - \tfrac{1}{2} rr'' \sin (a'' - a).$$

Substituting for r, r', and r'' their values, this becomes

$$\frac{a^2}{2} \left\{ \frac{\sin (a' - a)}{\cos^2 a'' \cos a \cos a'} + \frac{\sin (a'' - a')}{\cos^2 a \cos a' \cos a''} - \frac{\sin (a'' - a)}{\cos^2 a' \cos a \cos a''} \right\}.$$

Expanding the sines and making obvious reductions, we get

$$\frac{a^2}{2} \left\{ \frac{\tan a' - \tan a}{\cos^2 a''} + \frac{\tan a'' - \tan a'}{\cos^2 a} + \frac{\tan a - \tan a''}{\cos^2 a'} \right\};$$

or grouping differently, and putting $\sec^2 a$ for $\dfrac{1}{\cos^2 a}$, and so on,

$$\frac{a^2}{2} \left\{ \tan a \left(\sec^2 a' - \sec^2 a'' \right) + \tan a' \left(\sec^2 a'' - \sec^2 a \right) \right.$$
$$\left. + \tan a'' \left(\sec^2 a - \sec^2 a' \right) \right\}.$$

Lastly, putting $1 + \tan^2 a = 1 + m^2$ for $\sec^2 a$, and so on, we find the area of the triangle to be

$$\frac{a^2}{2} \left\{ m \left(m'^2 - m''^2 \right) + m' \left(m''^2 - m^2 \right) + m'' \left(m^2 - m'^2 \right) \right\},$$

which is quite symmetrical with respect to m, m', m''.

It will be easily seen, that the sides of the triangle are respectively

$$a \frac{m'' - m'}{\cos a}, \quad a \frac{m - m''}{\cos a'}, \quad a \frac{m' - m}{\cos a''}.$$

If these be called p, p', p'', and if ρ be the radius of the circle, by reduction, we obtain

$$p = 2\rho \sin (a'' - a'), \quad p' = 2\rho \sin (a - a''), \quad p'' = 2\rho \sin (a' - a).$$

If the values of the sines derived from these equations be substituted in the first expression for the area, it becomes

$$\frac{a}{2} \left(\frac{p}{\cos a} + \frac{p'}{\cos a'} + \frac{p''}{\cos a''} \right).$$

<div align="right">R. L. E.</div>

III.—INVESTIGATION OF THE GENERAL TERM OF THE EXPANSION OF THE TRUE ANOMALY IN TERMS OF THE MEAN.

THE equations, by means of which the true anomaly θ is to be determined in terms of the mean nt, are

$$nt = u - e \sin u,$$

$$\tan \frac{\theta - \omega}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{u}{2}.$$

Put $nt = z$, and let θ' be the value of θ when z is put for u; then

$$u = z + e \sin u,$$

and, by Lagrange's theorem,

$$\theta - \omega = \theta' - \omega + \frac{e}{1} \sin z \frac{d\theta'}{dz} + \frac{e^2}{1 . 2} \frac{d}{dz} \left\{ (\sin z)^2 \frac{d\theta'}{dz} \right\} + \dots$$

$$= \Sigma \frac{e^p}{1 . 2 \dots p} \frac{d^{p-1}}{dz^{p-1}} \left\{ (\sin z)^p \frac{d\theta'}{dz} \right\},$$

p being taken from 0 to ∞.

Now $\qquad \theta' - \omega = 2 \tan^{-1} \left(\sqrt{\frac{1 + e}{1 - e}} \tan \frac{z}{2} \right);$

$$\therefore \frac{d\theta'}{dz} = \frac{\sqrt{\dfrac{1 + e}{1 - e}} \left(\sec \dfrac{z}{2} \right)^2}{1 + \dfrac{1 + e}{1 - e} \left(\tan \dfrac{z}{2} \right)^2} = \frac{\sqrt{1 - e^2}}{1 - e \cos z},$$

which may be expanded in the series

$$l + 2\lambda \cos z + 2\lambda^2 \cos 2z + 2\lambda^3 \cos 3z + \dots$$

where $\lambda = \dfrac{1 - \sqrt{1 - e^2}}{e} = \dfrac{e}{2} + \dfrac{e^3}{8} + \dfrac{e^5}{16} + \dots$

$$\therefore \frac{d\theta'}{dz} = 2\Sigma\lambda \cos^m mz,$$

if the term corresponding to $m = 0$ be divided by 2.

A general expression for $(\sin z)^p$ is required, in terms of sines or cosines of multiples of z. Assume

$$\cos z + \sqrt{-1} \sin z = x, \quad \cos z - \sqrt{-1} \sin z = y,$$

then

$$(2\sqrt{-1} \sin z)^p = (x-y)^p = x^p - \frac{p}{1} x^{p-1} y + \frac{p(p-1)}{1.2} x^{p-2} y^2 - \ldots$$

but $xy = 1$; therefore $x^{p-1}y = x^{p-2}$, $x^{p-2}y^2 = x^{p-4}$, &c.;

$$\therefore (2\sqrt{-1} \sin z)^p = x^p - \frac{p}{1} x^{p-2} + \frac{p(p-1)}{1.2} x^{p-4} - \ldots$$

$$= \cos pz - \frac{p}{1} \cos (p-2) z + \frac{p(p-1)}{1.2} \cos (p-4) z - \ldots$$

$$+ \sqrt{-1}\left\{ \sin pz - \frac{p}{1} \sin (p-2) z + \frac{p(p-1)}{1.2} \sin (p-4) z - \ldots \right\},$$

and $\sqrt{-1} = \cos (4\mu + 1)\frac{\pi}{2} + \sqrt{-1} \sin (4\mu + 1)\frac{\pi}{2}$;

$$\therefore (\sqrt{-1})^p = \cos (4\mu + 1) p\frac{\pi}{2} + \sqrt{-1} \sin (4\mu + 1) p\frac{\pi}{2}.$$

When p is an integer, $(\sqrt{-1})^p$ will have only one value, therefore we may take $\mu = 0$;

$$\therefore (\sqrt{-1})^p = \cos p\frac{\pi}{2} + \sqrt{-1} \sin p\frac{\pi}{2}.$$

$$\therefore (\sin z)^p = \frac{1}{2^p}\left(\cos p\frac{\pi}{2} + \sqrt{-1} \sin p\frac{\pi}{2} \right)\left\{ \cos pz - \frac{p}{1} \cos (p-2) z \right.$$

$$+ \frac{p(p-1)}{1.2} \cos (p-4) z - \ldots$$

$$+ \sqrt{-1}\left\{ \sin pz - \frac{p}{1} \sin (p-2)z + \frac{p(p-1)}{1.2} \sin (p-4) z - \ldots \right\} \Big\}.$$

When p is an integer, $(\sin z)^p$ must be real; therefore taking only the real part of the second side,

$$(\sin z)^p = \frac{1}{2^p}\left\{ \cos \left(pz - p\frac{\pi}{2} \right) - \frac{p}{1} \cos \left((p-2) z - p\frac{\pi}{2} \right) \right.$$

$$+ \frac{p(p-1)}{1.2} \cos \left((p-4) z - p\frac{\pi}{2} \right) - \ldots \Big\},$$

the series to be continued till it terminate of itself, which it will do, since p is an integer. We have therefore

$$(\sin z)^p = \frac{1}{2^p} \Sigma (-1)^q \frac{p(p-1)\ldots(p-q+1)}{1.2\ldots q} \cos\left\{ (p-2q)z - p\frac{\pi}{2} \right\},$$

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where q is to be taken from 0.   Hence

$$(\sin z)^p \frac{d\theta'}{dz}$$

$$=\frac{1}{2^p} \Sigma\Sigma(-1)^q . \frac{p(p-1)\ldots(p-q+1)}{1 . 2 \ldots q} \lambda^m 2\cos\left\{(p-2q)z-p\frac{\pi}{2}\right\}\cos mz$$

$$=\frac{1}{2^p} \Sigma \Sigma (-1)^q \frac{p(p-1)\ldots(p-q+1)}{1 . 2 \ldots q} \lambda^m \times$$

$$\times\left\{\cos\left((p-2q+m)z-p\frac{\pi}{2}\right)+\cos\left((p-2q-m)z-p\frac{\pi}{2}\right)\right\},$$

Now in general

$$\frac{d^p \cos n\phi}{d\varphi^p} = n^p \cos\left(n\phi + p\frac{\pi}{2}\right),$$

$$\therefore \frac{d^{p-1}}{dz^{p-1}}\left\{(\sin z)^p \frac{d\theta'}{dz}\right\}$$

$$=\frac{1}{2^p} \Sigma \Sigma (-1)^q \frac{p(p-1)\ldots(p-q+1)}{1 . 2 \ldots q} \lambda^m \times$$

$$\times \{(p-2q+m)^{p-1}\sin(p-2q+m)z+(p-2q-m)^{p-1}\sin(p-2q-m)z\}.$$

Put $(p - 2q + m) = r$, and let us investigate the total coefficient of $r^{p-1} \sin rz$, when m and q vary from 0 upwards. Since $m = r - p + 2q$, when $q = 0$, $m = r - p$, so that the different values of

$$(-1)^q \frac{p(p-1)\ldots(p-q+1)}{1 . 2 \ldots q} \lambda^m$$

give the series

$$\lambda^{r-p} + \frac{p}{1} \lambda^{r-p+2} + \frac{p(p-1)}{1 . 2} \lambda^{r-p+4} \ldots$$

$$= \lambda^{r-p} (1 - \lambda^2)^p = \lambda^r (\lambda^{-1} - \lambda)^p,$$

in the case where r is greater than p.   But if r be less than p, since m must not be negative, q must begin from $\frac{p - r}{2}$ or $\frac{p - r + 1}{2}$, according as $p - r$ is even or odd, and m will begin from 0 or 1.   The greatest value of q will be p, since for higher values the coefficient

$$\frac{p(p - 1)\ldots(p - q + 1)}{1 . 2 \ldots q}$$

will vanish.   The corresponding value of m is $r + p$, so that the coefficient is in this case, beginning with the greatest values of m and q,

$$(-1)^p \left\{\lambda^{r+p} - \frac{p}{1} \lambda^{r+p-2} + \frac{p(p-1)}{1 . 2} \lambda^{r+p-4} - \ldots\right\},$$

continued as long as the index of λ does not become negative. We may write it

$$(-1)^p \lambda^{r+p} (1 - \lambda^{-2})^p = \lambda^r (\lambda^{-1} - \lambda)^p,$$

if we observe that negative powers of λ are to be rejected. Also, the term independent of λ, when there is one, must be divided by 2, because it arises from the term corresponding to $m = 0$ in $\dfrac{d\theta'}{dz}$.

Next, put $p - 2q - m = r$, then $m = p - r - 2q$. Here p must be greater than r, otherwise m would be negative. We obtain, as before, the series

$$\lambda^{p-r} - \frac{p}{1} \lambda^{p-r-2} + \frac{p(p-1)}{1 \cdot 2} \lambda^{p-r-4} - \dots$$
$$= \lambda^{p-r} (1 - \lambda^{-2})^p = (-1)^p \lambda^{-r} (\lambda^{-1} - \lambda)^p,$$

with the same restriction as before. Hence, when $p > r$, we have for the multiplier of $r^{p-1} \sin rz$,

$$\{\lambda^r + (-1)^p \lambda^{-r}\} (\lambda^{-1} - \lambda)^p.$$

We have seen that when $p < r$, part only of this formula is required. But the same expression may be used in both cases, because when $p < r$, $\lambda^{-r} (\lambda^{-1} - \lambda)^p$ will contain only negative powers of λ, and is therefore to be rejected entirely. Hence the term in $\theta - \omega$, involving $\sin rnt$, is

$$\Sigma \frac{e^p}{1 \cdot 2 \, .. \, p} \frac{1}{2^p} \{\lambda^r + (-1)^p \lambda^{-r}\} (\lambda^{-1} - \lambda)^p r^{p-1} \sin rnt$$

$$= \left\{ \lambda^r \Sigma \frac{\left(\frac{re}{2}\right)^p (\lambda^{-1} - \lambda)^p}{1 \cdot 2 \dots p} + \lambda^{-r} \Sigma \frac{\left(-\frac{re}{2}\right)^p (\lambda^{-1} - \lambda)^p}{1 \cdot 2 \dots p} \right\} \frac{\sin rnt}{r}$$

$$= \left\{ \lambda^r \varepsilon^{\frac{re}{2}(\lambda^{-1} - \lambda)} + \lambda^{-r} \varepsilon^{-\frac{re}{2}(\lambda^{-1} - \lambda)} \right\} \frac{\sin rnt}{r}.$$

Here r may have any value, positive or negative. The terms arising from negative values of r are identical with those from equal positive values; and therefore if r have positive values only, including 0,

$$\theta - \omega = 2\Sigma \left\{ \lambda^r \varepsilon^{\frac{re}{2}(\lambda^{-1} - \lambda)} + \lambda^{-r} \varepsilon^{-\frac{re}{2}(\lambda^{-1} - \lambda)} \right\} \frac{\sin rnt}{r}.$$

This expression must be applied only by developing it in the form in which it stands, dividing by 2 the terms in that development which do not involve λ, and rejecting all negative powers of that quantity.

<div align="right">S. S. G.</div>

IV.—DEMONSTRATIONS OF THEOREMS IN THE DIFFERENTIAL CALCULUS AND CALCULUS OF FINITE DIFFERENCES.

I propose in this Article to bring together the more important of the theorems in the Differential Calculus and in the Calculus of Finite Differences, which, depending on one common principle, can be proved by the method of the separation of symbols. These theorems are usually demonstrated by induction in each particular case, which, although a method satisfactory so far as it goes, wants that generality which is desirable in Analytical Demonstrations. As the ordinary Binomial Theorem is the basis on which these theorems are founded, it will be not amiss to say a few words by way of preface regarding the extent of its application, which being said once for all, will prevent useless repetition when we treat of each particular case.

The theorem that

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}b^3 + \&c.$$

is originally proved when a and b are numbers, and $(a+b)^n$ represents the repetition of the operation n times, implying that n is an integer number. Having the form of the expansion once suggested, it can be shown, by the method of Euler, that the same form is true when n is a fraction or negative number; in which case the left-hand side of the equation acquires different meanings. Moreover, it will be found, on examining Euler's demonstration, that it includes not only these cases, but also all those in which a, b, and n are operations subject to certain laws; for it may be seen, that in the proof no other properties are presumed than that a, b, and n are distributive and commutative functions, and that a^n, b^n are subject to the laws of index functions. These laws are,

(1) The commutative, $ab = ba$,

(2) The distributive, $c(a + b) = ca + cb$,

(3) The index law, $a^m \cdot (a^n) = a^{m+n}$.

Now, since it can be shown that the operations both in the Differential Calculus and the Calculus of Finite Differences are subject to these laws, the Binomial Theorem may be at once assumed as true with respect to them, so that it is not necessary to repeat the demonstration of it for each case.* This being premised, I proceed to consider the particular cases of the applications of these theorems.

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* It is scarcely necessary to add, that those theorems which depend on the binomial, as the polynomial and exponential, are equally extensive, so that they too may be applied to the Differential Calculus and Calculus of Finite Differences.

1. If $u = f(x, y)$ be a function of two independent variables, we find that

$$d(u) = \frac{du}{dx} dx + \frac{du}{dy} dy = \left(\frac{d}{dx} dx + \frac{d}{dy} dy \right) u,$$

by separating the symbols. Now, if we wish to find the nth differential of a function of two variables, we have merely, by the principle of indices, to affix the index n to the sign of operation on both sides, when we get

$$d^n(u) = \left(\frac{d}{dx} dx + \frac{d}{dy} dy \right)^n u.$$

Now the operation on the second side, being a binomial raised to a power, may, by what has just been said, be expanded by the binomial theorem, so that we have

$$d^n u = \frac{d^n u}{dx^n} dx^n + n \frac{d^{n-1} u}{dx^{n-1}} \frac{du}{dy} + \frac{n(n-1)}{1 \cdot 2} \frac{d^{n-2} u}{dx^{n-2}} \frac{d^2 u}{dy^2} - \&c.$$

This theorem, which can be proved by induction only for positive integer powers of n, that is, for cases of ordinary differentiation, is shown by this method to be true when n is fractional or negative, that is, in the cases of integration and general differentiation.

If we suppose u to be a function of three or more variables, we might by means of the polynomial theorem, expand $d^n \cdot (u)$; but it is not necessary to dwell upon the result, as there is little interest attached to it.

2. I shall next proceed to the elegant theorem of Leibnitz, for finding the nth differential of the product of two functions, a theorem which, when generalized, is most fertile in consequences.

Let u, v be the two functions. Then

$$\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}.$$

This may be put under the form

$$\frac{d}{dx} (u\, v) = \left(\frac{d'}{dx} + \frac{d}{dx} \right) u\, v,$$

if we agree to represent by $\dfrac{d'}{dx}$ an operation which acts on v, but

not on u, and by $\dfrac{d}{dx}$ an operation which acts on u and not on v.

These operations from their nature are distributive, and as they are independent of each other, they must be commutative; hence they come under the circumstances to which the binomial theorem applies. Taking then the nth differential,

$$\left(\frac{d}{dx} \right)^n (u\, v) = \left(\frac{d}{dx} + \frac{d}{dx} \right)^n u\, v$$

$$= \left\{ \left(\frac{d'}{dx} \right)^n + n \left(\frac{d'}{dx} \right)^{n-1} \frac{d}{dx} + \frac{n \cdot n-1}{1 \cdot 2} \left(\frac{d'}{dx} \right)^{n-2} \left(\frac{d}{dx} \right)^2 + \&c. \right\} u\, v;$$

or applying the operations directly to the quantities which they affect,

$$= u \frac{d^n v}{dx^n} + n \frac{du}{dx} \frac{d^{n-1} v}{dx^{n-1}} + \frac{n(n-1)}{1 \cdot 2} \frac{d^2 u}{dx^2} \frac{d^{n-2} v}{dx^{n-2}} + \&\text{c.}$$

This theorem is true, like the former, when n is negative or fractional. In the former case, the form is the same as the series at which we arrive by integration by parts, which we thus see to be a particular case of the theorem of Leibnitz.

3. In this expression, when n is negative, let $v = 1$. Then

$$\frac{d^{-n} v}{dx^{-n}} = \frac{x^n}{n!}, \quad \frac{d^{-(n+1)} v}{dx^{-(n+1)}} = \frac{x^{n+1}}{(n+1)!}, \quad \&\text{c.}$$

so that

$$\left(\frac{d}{dx}\right)^{-n} u = \int^n dx^n \, u = \frac{x^n}{n!} \, u - n \frac{x^{n+1}}{(n+1)!} \frac{du}{dx}$$
$$+ \frac{n(n+1)}{1 \cdot 2} \frac{x^{n+2}}{(n+2)!} \frac{d^2 u}{dx^2} + \&\text{c.}$$
$$= \frac{x^{n-1}}{(n-1)!} \left(\frac{x}{n} u - \frac{x^2}{n+1} \frac{du}{dx} + \frac{1}{1 \cdot 2} \frac{x^3}{n+2} \frac{d^2 u}{dx^2} - \&\text{c.}\right),$$

which is the general expression for the nth integral of any function.

4. In this last formula, if we make $n = 1$ when it becomes a simple integral, we have

$$\int dx \, u = xu - \frac{x^2}{1 \cdot 2} \frac{du}{dx} + \frac{x^3}{1 \cdot 2 \cdot 3} \frac{d^2 u}{dx^2} - \&\text{c.},$$

the well known series of Bernouilli; which thus appears to be also a particular case of the theorem of Leibnitz when extended to general indices.

5. In this theorem, let us suppose $v = \varepsilon^{ax}$; then as $\frac{dv}{dx} = a\varepsilon^{ax}$

$= av$, we have $\frac{d}{dx} = a$, and therefore

$$\frac{d^n}{dx^n} (\varepsilon^{ax} u) = \left(a + \frac{d}{dx}\right)^n u \, \varepsilon^{ax};$$

whence $\left(a + \frac{d}{dx}\right)^n u = \varepsilon^{-ax} \left(\frac{d}{dx}\right)^n \varepsilon^{ax} u,$

which is the theorem given in Art. V. of our first Number.

6. In the Calculus of Finite Differences there are more theorems than in the Differential Calculus depending on the expansion of a binomial, in consequence of the relation which subsists between two kinds of operations, that of taking the increment and that of taking the difference. It is not usual to use a separate symbol for the former, but in Art. II. of the second Number of this Journal, I adopted the symbol D to represent this operation, as it simplified

greatly the expressions. For the same reason I shall continue to employ it, and I hope that its utility will compensate for any disadvantage which may accrue from using a new notation. Before proceeding, I will say a few words concerning the operation represented by this symbol D. Its definition is, that

$$\mathrm{D}f(x) = f(x + 1).$$

Now we know by Taylor's theorem that

$$\varepsilon^{h\frac{d}{dx}} f(x) = f(x + h),$$

whatever h may be; making $h = 1$, we have

$$\varepsilon^{\frac{d}{dx}} = f(x + 1).$$

Consequently $\mathrm{D} = \varepsilon^{\frac{d}{dx}}$;

from which we see that $\mathrm{D}^h f(x) = f(x + h)$, whatever h may be.

Also, since

$$\Delta f(x) = f(x + 1) - f(x) = \mathrm{D}f(x) - f(x) = (\mathrm{D} - 1)f(x),$$

we have $\Delta = \mathrm{D} - 1$ and $\mathrm{D} = 1 + \Delta$.

Consequently, D being a linear compound of commutative and distributive operations, is also a commutative and distributive operation. It is also subject to the laws of index functions,

since $\mathrm{D}^k\mathrm{D}^h f(x) = \mathrm{D}^k f(x + h) = f(x + h + k) = \mathrm{D}^{h+k} f(x).$

7. This being premised, since we have

$$\mathrm{D}u_x = (1 + \Delta) u_x,$$
$$\mathrm{D}^n u_x = (1 + \Delta)^n u_x;$$

and by the binomial theorem,

$$\mathrm{D}^n u_x = \left\{1 + n\Delta + \frac{n(n-1)}{1 \cdot 2} \Delta^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \Delta^3 + \&c.\right\} u_x,$$

or $u_{x+n} = u_x + n\Delta u_x + \dfrac{n(n-1)}{1 \cdot 2} \Delta^2 u_x + \&c.$

Again, $\mathrm{D}^n u_x = (\mathrm{D}^{-1})^{-n} u_x = \{\mathrm{D}^{-1}(\mathrm{D} - \Delta)\}^{-n} u_x$
$= (1 - \Delta\mathrm{D}^{-1})^{-n} u_x;$

whence by the binomial theorem

$$\mathrm{D}^n u_x = \left(1 + n\Delta\mathrm{D}^{-1} + \frac{n(n+1)}{1 \cdot 2} \Delta^2 \mathrm{D}^{-2}\right.$$
$$\left. + \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} \Delta^3 \mathrm{D}^{-3} + \&c.\right) u_x,$$

and therefore

$$u_{x+n} = u_x + n\Delta u_{x-1} + \frac{n(n+1)}{1 \cdot 2} \Delta u_{x-2} + \&c.$$

8. Besides these there are two other expressions for u_{x+n}, which, though not depending on the binomial theorem, are founded on the same principles.

$$u_{x+n} - u_x = (D^n - 1) u_x$$
$$= \Delta (D - 1)^{-1} (D^n - 1) u_x,$$

since $\Delta = D - 1$.

Expanding in the same way as we would expand $\dfrac{x^n - 1}{x - 1}$, we find

$$u_{x+n} - u_x = \Delta (1 + D + D^2 + \&c. + D^{n-1}) u_x,$$

and therefore

$$u_{x+n} = u_x + \Delta u_x + \Delta u_{x+1} + \Delta u_{x+2} + \&c. + \Delta u_{x+n-1}.$$

Similarly,

$$u_{x+n} - \Delta^n u_x = (D^n - \Delta^n) u_x = (D - \Delta)^{-1} (D^n - \Delta^n) u_x,$$

since $D - \Delta = 1$.

Therefore, expanding as we would expand $\dfrac{x^n - a^n}{x - a}$, we find

$$u_{x+n} - \Delta^n u_x = (D^{n-1} + \Delta D^{n-2} + \Delta^2 D^{n-3} + \&c. + \Delta^{n-2} D + \Delta^{n-1}) u_x,$$

and therefore

$$u_{x+n} = u_{x+n-1} + \Delta u_{x+n-2} + \&c. + \Delta^{n-2} u_{x+1} + \Delta^{n-1} u_x + \Delta^n u_x.$$

9. Corresponding to these theorems for $D^n u_x$, we have theorems for $\Delta^n u_x$.

Since $\Delta u_x = (D - 1) u_x$, $\Delta^n u_x = (D - 1)^n u_x$, and therefore by the binomial theorem

$$\Delta^n u_x = \left(D^n - n D^{n-1} + \frac{n(n-1)}{1 \cdot 2} D^{n-2} - \&c. \right) u_x$$

$$= u_{x+n} - n u_{x+n-1} + \frac{n(n-1)}{1 \cdot 2} u_{x+n-2} - \&c.$$

Again, $\Delta^r u_{x+n} = \Delta^r D^r u_{x+n-r} = \Delta^r \{D^{-1} (D - \Delta)\}^{-r} u_{x+n-r},$

$$= \Delta^r (1 - \Delta D^{-1})^{-r} u_{x+n-r},$$

and expanding

$$\Delta^r u_{x+n} = \Delta^r \left\{ 1 + r \Delta D^{-1} + \frac{r(r+1)}{1.2} \Delta^2 D^{-2} + .. \right\} u_{x+n-r}$$

$$= \Delta^r u_{x+n-r} + r \Delta^{r+1} u_{x+n-r-1}$$

$$+ \frac{r(r+1)}{1.2} \Delta^{r+2} u_{x+n-r-2} \mp \cdots$$

10. Connected with this subject is a formula for the transformation of series, which is useful for the purpose of changing diverging into converging series. The proof of this, which is usually made to depend on the theory of generating functions, can be much more simply derived from the theory I am here develop-

ing, and the same may be said of all theorems usually demonstrated by generating functions. Let the given series be

$$S = y_x + y_{x+1} + y_{x+2} + y_{x+3} + \cdots$$
$$= (1 + D + D^2 + D^3 + \ldots) y_x = (1 - D)^{-1} y_x;$$

and let it be desired to change this into one depending on

$$ay_x + a_1 y_{x+1} + a_2 y_{x+2} + \cdots + a_n y_{x+n}$$
$$= (a + a_1 D + a_2 D^2 + \cdots + a_n D^n) y_x = \nabla y_x$$

if we put $\quad \nabla = a + a_1 D + a_2 D^2 + \cdots + a_n D^n.$

Now it is to be observed, that any algebraic combination of the symbols D and Δ with constants will be likewise subject to the same laws of combination as these symbols, and may therefore be treated in the same way.

Hence, making $a + a_1 + a_2 + \ldots + a_n = K$, we shall have

$$S = (1 - D)^{-1} y_x = (K - \nabla)^{-1} (K - \nabla) (1 - D)^{-1} y_x,$$

since the operations $(K - \nabla)^{-1}$, $K - \nabla$ destroying each other, do not affect the equation. Now

$$(1 - D)^{-1}(K - \nabla) = (1 - D)^{-1} \{ a_1(1 - D) + a_2(1 - D^2)$$
$$+ a_3(1 - D^3) + \ldots + a_n(1 - D^n)$$
$$= a_1 + a_2(1 + D) + a_3(1 + D + D^2) + \cdots$$
$$+ a_n(1 + D + D^2 + \cdots + D^{n-1})$$
$$= a_1 + a_2 + a_3 + \cdots + a_n$$
$$+ (a_2 + a_3 + \cdots + a_n) D$$
$$+ (a_3 + \cdots + a_n) D^2$$
$$+ \cdots + \cdots$$

And therefore

$$S = (a_1 + a_2 + a_3 + \cdots + a_n) (K - \nabla)^{-1} y_x$$
$$+ (a_2 + a_3 + \cdots + a_n) (K - \nabla)^{-1} y_{x+1}$$
$$+ (a_3 + \cdots + a_n) (K - \nabla)^{-1} y_{x+2}$$
$$+ \cdots + \cdots$$

and expanding $(K - \nabla)^{-1}$, we find

$$S = (a_1 + a_2 + a_3 + \ldots + a_n) \left(\frac{y_x}{K} + \frac{\nabla y_x}{K^2} + \frac{\nabla^2 y_x}{K^3} + \cdots \right)$$
$$+ (a_2 + a_3 + \ldots + a_n) \left(\frac{y_{x+1}}{K} + \frac{\nabla y_{x+1}}{K^2} + \frac{\nabla^2 y_{x+1}}{K^3} + \ldots \right)$$
$$+ (a_3 + \ldots + a_n) \left(\frac{y_{x+2}}{K} + \frac{\nabla y_{x+2}}{K^2} + \frac{\nabla^2 y_{x+2}}{K^3} + \ldots \right),$$

which is the required transformation for S.

11. In the particular case where

$$S = ax + a_1 x^2 + a_2 x^3 + a_3 x^4 + \ldots,$$

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Euler has employed a very elegant method of transformation, the reason for which appears very clearly, if we follow the method of the separation of symbols.

Let us suppose a, a_1, a_2, a_3, ... to be terms of a series which can be derived one from the other by a certain law, so that $a_1 = Da$, $a_2 = D^2 a$, and so on. Then

$$S = (x + x^2 D + x^3 D^2 + ...) a$$
$$= x (1 - xD)^{-1} a = x (1 - x - x\Delta)^{-1} a$$
$$= \frac{x}{1 - x} \left(1 - \frac{x}{1 - x} \Delta\right)^{-1} a$$
$$= \frac{x}{1 - x} \left(1 + \frac{x}{1 - x} \Delta + \frac{x^2}{(1 - x)^2} \Delta^2 + ...\right) a$$
$$= \frac{ax}{1 - x} + \Delta a \frac{x^2}{(1 - x)^2} + \Delta^2 a \frac{x^3}{(1 - x)^3} + ...$$

12. The expression for the total difference of a function of two variables, in terms of the partial differences, is not so simple as its analogue in the Differential Calculus. If we represent the total difference by Δ, and the partial differences with respect to x and y by Δ_x, Δ_y, and the corresponding increments by D_x, D_y, we have, since

$$\Delta_x u_{x,\, y+1} = u_{x+1,\, y+1} - u_{x,\, y+1},$$
$$\text{and } \Delta_y u_{x+1,\, y} = u_{x+1,\, y+1} - u_{x+1,\, y},$$

by adding them together, and subtracting from

$$2\Delta u_{x,\, y} = 2 (u_{x+1,\, y+1} - u_{x,\, y}),$$
$$2\Delta u_{x,\, y} - \Delta_x u_{x,\, y+1} - \Delta_y u_{y,\, x+1} = u_{x,\, y+1} - u_{x,\, y} + u_{x+1,\, y} - u_{x,\, y}.$$
$$\text{whence } 2\Delta u_{x,\, y} = \Delta_y D_x u_{x,\, y} + \Delta_x D_y u_{x,\, y} + \Delta_y u_{x,\, y} + \Delta_x u_{x,\, y},$$
$$\text{or } \Delta u_{x,\, y} = \tfrac{1}{2} \{\Delta_y (1 + D_x) + \Delta_x (1 + \Delta_y)\} u_{x,\, y}.$$

Now, since all the symbols are relatively commutative, inasmuch as $D . \Delta u_x = \Delta . Du_x$ when they refer to the same variable, and as when referring to different variables they are wholly independent, and therefore commutative; and since all the symbols are also distributive, the binomial theorem may be here applied, and therefore

$$\Delta^n u_{x,\, y} = \frac{1}{2^n} \{\Delta_y (1 + D_x) + \Delta_x (1 + D_y)\}^n u_{x,\, y}$$

$$= \frac{1}{2^n} \{\Delta_y{}^n (1 + D_x)^n + n \Delta_y{}^{n-1} (1 + D_x)^{n-1} \Delta_x (1 + D_y)$$

$$+ \frac{n (n - 1)}{1 . 2} \Delta_y{}^{n-2} (1 + D_x)^{n-2} \Delta_x{}^2 (1 + D_y)^3 + ...\} u_{x,\, y}.$$

If each term be expanded, and the operations indicated by D_x, D_y be effected, we shall obtain a result in Δ_x and Δ_y; but it is so complicated, that it is better to keep it in the unexpanded form, as we thus see the law of formation more distinctly.

13. It is also obvious, that as

$$\Delta u_{x,y} = (D - 1) u_{x,y},$$

when Δ and D are total operations referring to both variables,

$$\Delta^n u_{x,y} = (D^n - n D^{n-1} + \frac{n(n-1)}{1.2} D^{n-2} - \ldots) u_{x,y}$$

$$= u_{x+n,\,y+n} - n u_{x+n-1,\,y+n-1} + \cdots$$

and similarly

$$D^n u_{x,y} = u_{x,y} + n\Delta u_{x,y} + \frac{n(n-1)}{1.2} \Delta^2 u_{x,y} + \cdots$$

14. We shall proceed now to the operations on the products of two or more functions of the same variable.

$$\Delta u_x v_x = u_{x+1} v_{x+1} - u_x v_x = (D D_1 - 1) u_x v_x,$$

where we suppose D to refer to u_x and D_1 to v_x. Therefore

$$\Delta^n u_x v_x = (D D_1 - 1)^n u_x v_x$$

$$= \left(D^n D_1{}^n - n D^{n-1} D_1{}^{n-1} + \frac{n(n-1)}{1.2} D^{n-2} D_1{}^{n-2} - \ldots \right) u_x v_x;$$

therefore $\Delta^n u_x v_x = u_{x+n} v_{x+n} - n u_{x+n-1} v_{x+n-1} + \cdots$

This is true whatever n is. Let it be negative, then

$$\Delta^{-n} u_x v_x = \Sigma^n u_x v_x = u_{x-n} v_{x-n} + n u_{x-n-1} v_{x-n-1} + \cdots$$

15. The nth difference of the product of two functions may be expanded in another manner. Since

$$\Delta u_x v_x = u_{x+1} v_{x+1} - u_x v_x$$

$$= u_x \Delta v_x + v_{x+1} \Delta u_x$$

$$= (\Delta + D\Delta_1) u_x v_x,$$

(where Δ, D refer to u_x and Δ_1 to v_x),

$$\Delta^n u_x v_x = (\Delta + D\Delta_1)^n u_x v_x$$

$$= \left(\Delta^n + \Delta^{n-1} \Delta_1 D + \frac{n(n-1)}{1.2} \Delta^{n-2} \Delta_1{}^2 D^2 + \ldots \right) u_x v_x;$$

therefore

$$\Delta^n u_x v_x = v_x \Delta^n u_x + n\Delta v_x \Delta^{n-1} u_{x+1} + \frac{n(n-1)}{1.2} \Delta^2 v_x \Delta^{n-2} u_{x+2} + \cdots$$

When n becomes negative, $\Delta^{-n} u_x v_x = \Sigma^n u_x v_x$, and therefore

$$\Sigma^n (u_x v_x) = v_x \Sigma^n u_x - n \Delta v_x \Sigma^{n+1} u_{x+1} + \frac{n(n+1)}{1.2} \Delta^2 v_x \Sigma^{n+2} u_{x+2} - \cdots$$

which is the formula for integration by parts; and when $n = 1$,

$$\Sigma (u_x v_x) = v_x \Sigma u_x - \Delta v_x \Sigma^2 u_{x+1} + \Delta^2 v_x \Sigma^3 u_{x+2} - \cdots$$

16. If we suppose $u_x = x^0 = 1$ in the former expression, we have, since $u_x = u_{x+1} = u_{x+2}$, &c.

$$\Sigma^n u_x = \Sigma^n . 1 = \frac{x(x+1)\ldots(x+n-1)}{(n)!},$$

and therefore

$$\Sigma^n(v_x) = \frac{x(x+1)\dots(x+n-1)}{(n-1)!}\left\{\frac{v_x}{n} - \frac{(x+n)}{1}\frac{\Delta v_x}{n+1}\right.$$
$$\left. + \frac{(x+n)(x+n+1)}{1.2}\frac{\Delta^2 v_x}{n+2} - \dots\right\},$$

which, when $n = 1$, becomes

$$\Sigma(v_x) = xv_x - \frac{x(x+1)}{1.2}\Delta v_x + \frac{x(x+1)(x+2)}{1.2.3}\Delta^2 v_x - \dots$$

a series which bears a close analogy with that of Bernouilli in the Integral Calculus.

17. Again, since

$$\Delta^n u_x v_x = (DD_1 - 1)^n u_x v_x,$$

if we make $v_x = a^x$, $D_1 a^x = a^{x+1} = a.a^x$, so that $D_1 = a$, and

$$\Delta^n u_x a^x = (Da - 1)^n u_x a^x,$$

and $(Da - 1)^n u_x = a^{-x}.\Delta^n(u_x a^x)$.

If for a we put $\dfrac{1}{a}$, we obtain

$$(D - a)^n u_x = a^{x+n}\Delta^n(u_x a^{-x}),$$

which is the theorem given in Article II. of the second Number of this Journal.

18. The connexion which exists between the Differential Calculus and the Calculus of Finite Differences, gives rise to various elegant theorems; the first of which is the celebrated theorem of Lagrange, that

$$\Delta^n u_x = (\varepsilon^{\frac{d}{dx}} - 1)^n u_x.$$

For as we have

$$\Delta u_x = (D - 1)u_x = (\varepsilon^{\frac{d}{dx}} - 1)u_x,$$

raising the symbol of operation to the nth power on each side,

$$\Delta^n u_x = (\varepsilon^{\frac{d}{dx}} - 1)^n u_x.$$

It is usual to make the proof of this theorem a matter of some difficulty, but it follows at once from the theory of the laws of combination of the symbols. It is true whatever n may be, and therefore when n is negative, or

$$\Sigma^n u_x = (\varepsilon^{\frac{d}{dx}} - 1)^{-n} u_x,$$

or for the particular value 1 of n,

$$\Sigma u_x = (\varepsilon^{\frac{d}{dx}} - 1)^{-1} u_x.$$

The second side, when expanded by the numbers of Bernouilli, gives

$$\Sigma u_x = \left\{ \left(\frac{d}{dx}\right)^{-1} - \frac{1}{2} + \frac{B_1}{1 \cdot 2} \frac{d}{dx} - \frac{B_3}{1 \cdot 2 \cdot 3 \cdot 4} \frac{d^3}{dx^3} + \dots \right\} u_x$$

$$= \int u_x dx - \frac{u_x}{2} + \frac{B_1}{1 \cdot 2} \frac{du_x}{dx} - \frac{B_3}{1 \cdot 2 \quad 3 \cdot 4} \frac{d^3 u_x}{dx^3} + \dots$$

19. The theorem for expressing the nth difference in terms of the nth and higher differential coefficients, may be derived very readily without expansion from the fundamental theorem

$$\Delta u_x = (\varepsilon^{\frac{d}{dx}} - 1)\, u_x;$$

for we shall also have

$$\Delta\, (\varepsilon^{h\frac{d}{dx}} u) = \varepsilon^{h\frac{d}{dx}}(\varepsilon^{\frac{d}{dx}} - 1)\, u_x, \text{ when } h = 0.$$

But $\varepsilon^{h\frac{d}{dx}}(\varepsilon^{\frac{d}{dx}} - 1)$ is the difference of $\varepsilon^{h\frac{d}{dx}}$, taken with respect to h, and may be represented by $\Delta_h\, \varepsilon^{h\frac{d}{dx}}$, where Δ_h implies that the sign of operation affects h only. Hence we have

$$\Delta\, (\varepsilon^{h\frac{d}{dx}} u_x) = \Delta_h\, (\varepsilon^{h\frac{d}{dx}} u_x), \text{ when } h = 0.$$

By this artifice the symbol of operation is transferred from the x to the h. Now, taking the nth difference on both sides,

$$\Delta^n\, (\varepsilon^{h\frac{d}{dx}} u_x) = \Delta_h^{\,n}\, (\varepsilon^{h\frac{d}{dx}} u_x), \text{ when } h = 0.$$

On expanding $\varepsilon^{h\frac{d}{dx}}$ on the second side, and effecting the operation $\Delta_h^{\,n}$, it appears that all the terms will vanish till the $(n+1)^{\text{th}}$, so that replacing h by 0 we have the usual formula

$$\Delta^n u_x = \frac{\Delta^n 0^n}{n!} \frac{d^n u}{dx^n} + \frac{\Delta^n 0^{n+1}}{(n+1)!} \frac{d^{n+1} u}{dx^{n+1}} + \dots$$

20. The same method affords an easy proof of a theorem first given by Sir John Herschel in the *Phil. Trans.*, 1816, for expanding any function of ε^t.

$$f(\varepsilon^t) = f(\varepsilon^t)\, \varepsilon^{xt} \text{ when } x = 0.$$

And since $\varepsilon^{\frac{d}{dx}} \cdot \varepsilon^{xt} = \varepsilon^t$ when $x = 0$,

$$f(\varepsilon^t) = f(\varepsilon^{\frac{d}{dx}})\, \varepsilon^{xt} \text{ when } x = 0,$$

or expanding ε^{xt}, putting 0 for x and $1 + \Delta$ for $\varepsilon^{\frac{d}{dx}}$, we get

$$f(\varepsilon^t) = f(1 + \Delta)\, 1 + f(1 + \Delta) 0 \cdot t + f(1 + \Delta) \frac{0^2 t^2}{1 \cdot 2} + \dots$$

which is the form given by Sir John Herschel.

I cannot mention the name of this mathematician without correcting an error into which I fell in Article V. of the first

Number of this Journal. I there stated that, so far as I knew, Brisson was the first person who had applied the method of the separation of symbols to the solution of differential equations. I have since found that Sir John Herschel was really the first person who did so, in a paper published in the *Philosophical Transactions* for 1816, five years before the date of Brisson's Memoir. It is much to be regretted, that neither Sir John Herschel himself, nor any other person, followed up this method, which is calculated to be of so much use in the higher analysis. Perhaps this may have arisen from the theory of the method not having been properly laid down, so that a certain degree of doubt existed as to the correctness of the principle. I trust, however, that the various developments which I have given in several articles in this Journal, of the principles of the method as well as the proofs of its utility, are sufficient for removing all doubts on this head, and that it will now be regarded as a powerful instrument in the hands of mathematicians.

<div align="right">D. F. G.</div>

V.—ON THE CONDITIONS OF EQUILIBRIUM OF A RIGID SYSTEM, &c.*

SUPPOSE the forces P, P', P'', ... applied at the points (x, y, z), (x', y', z'), (x'', y'', z''), ... of any rigid system, and let a, β, γ, a', β', γ', ... be the angles which their directions make with the axes of coordinates.

Let AO be a line drawn from the origin A to any point O taken arbitrarily, and through O let a plane be drawn perpendicular to AO.

The effect of the forces will not be altered by removing their points of application to the points where their directions meet this plane. This being done, we may resolve each force into two; one in the plane, and coinciding in magnitude and direction with the projection of the force upon the plane, and the other perpendicular to it. We shall thus have two sets of forces; one set lying entirely in the assumed plane, and the other perpendicular to it, and parallel to each other; and it is plain that these two sets must be separately in equilibrium.

Now a condition of equilibrium for parallel forces is, that their sum shall equal nothing. And a condition for forces in one plane is, that the sum of their moments about any point in it shall equal nothing.

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* From a Correspondent.

To express these two conditions, let θ be the angle between the direction of the force P, and the line AO; then the two resolved parts of P will be $P \cos \theta$ perpendicular to the assumed plane, and $P \sin \theta$ in the plane. Also let q be the perpendicular drawn from O upon the direction of the latter force, *i.e.* upon the projection of P on the plane. And calling θ', q',... the corresponding quantities for P', P'',... the two conditions above mentioned will be

$$(1) \quad P \cos \theta + P' \cos \theta' + \ldots = 0,$$
$$(2) \quad Pq \sin \theta + P'q' \sin \theta' + \ldots = 0.$$

Let a, b, c be the cosines of the angles between the line AO and the three axes respectively, then

$$(3) \cos \theta = a \cos \alpha + b \cos \beta + c \cos \gamma,$$
$$(4) \sin^2 \theta = (b \cos \gamma - c \cos \beta)^2 + (c \cos \alpha - a \cos \gamma)^2$$
$$+ (a \cos \beta - b \cos \alpha)^2.$$

Hence, substituting for $\cos \theta$, $\cos \theta'$, ... in (1), we have

$$a (P \cos \alpha + P' \cos \alpha' + \ldots) + b (P \cos \beta + P' \cos \beta' + \ldots)$$
$$+ c (P \cos \gamma + P' \cos \gamma' + \ldots) = 0,$$
$$\text{or } aX + bY + cZ = 0,$$

and therefore, since any two of the cosines a, b, c are arbitrary, we must have separately

$$X = 0, \quad Y = 0, \quad Z = 0.$$

To express the condition (2). Let a plane be drawn containing the direction of the force P, and perpendicular to the assumed plane upon which we have projected the forces. The intersection of these two planes will evidently be the projection of P upon the latter, and a perpendicular drawn from the origin upon the former will be precisely equal to the perpendicular q. Suppose l, m, n are the cosines of the angles between this perpendicular and the three axes, then the equation to the plane in question will be

$$(5) \quad l (\xi - x) + m (\eta - y) + n (\zeta - z) = 0,$$

and therefore evidently

$$(6) \quad q = lx + my + nz.$$

Also l, m, n will be subject to the equations expressing the two conditions that the plane (5) contains the force P, and is perpendicular to the plane on which the forces are projected; these are evidently

$$l \cos \alpha + m \cos \beta + n \cos \gamma = 0$$
$$la + mb + nc = 0,$$

from which we deduce immediately

$$\frac{l}{b \cos \gamma - c \cos \beta} = \frac{m}{c \cos \alpha - a \cos \gamma} = \frac{n}{a \cos \beta - b \cos \alpha} = \frac{1}{\sin \theta}$$

(see equation (4); and the theorem at page 187).

Hence, substituting for l, m, n in (6), and multiplying by P, we obtain

$$\text{P}q \sin \theta = \text{P} (b \cos \gamma - c \cos \beta) \, x + \text{P} (c \cos \alpha - a \cos \gamma) \, y$$
$$+ \text{P} (a \cos \beta - b \cos \alpha) \, z.$$

This is the moment of the projection of P upon the arbitrarily assumed plane, with reference to the point O. If we form similar equations for the other forces and collect their sum, we find

$$\text{P}q \sin \theta + \text{P}'q' \sin \theta' + \ldots = a \cdot \{\text{P} (z \cos \beta - y \cos \gamma)$$
$$+ \text{P}' (z' \cos \beta' - y' \cos \gamma') + \ldots \}$$
$$+ b \cdot \{\text{P} (x \cos \gamma - z \cos \alpha) + \ldots \} + c \cdot \{\text{P}(y \cos \alpha - x \cos \beta) + \ldots \}.$$

The expression on the right of this equation, which we may write for shortness $a\text{L} + b\text{M} + c\text{N}$, must equal nothing in the case of equilibrium. Hence, as before, we must have separately

$$\text{L} = 0, \ \text{M} = 0, \ \text{N} = 0.$$

These equations, together with the three X = 0, Y = 0, Z = 0, are therefore necessary conditions of equilibrium, and it is easily seen that they are sufficient.

It appears from the preceding demonstration that the moment of the forces about any axis passing through the origin, and making with the axes of coordinates angles whose cosines are a, b, c, is

$$(7) \ a\text{L} + b\text{M} + c\text{N}.$$

To find the *principal moment*, we must make this expression a maximum.

Putting therefore $\text{L}da + \text{M}db + \text{N}dc = 0$, and combining this with the equation $ada + bdb + cdc = 0$, we find (since any two of the differentials are independent)

$$(8) \ \frac{\text{L}}{a} = \frac{\text{M}}{b} = \frac{\text{N}}{c} = \sqrt{\text{L}^2 + \text{M}^2 + \text{N}^2} = a\text{L} + b\text{M} + c\text{N} \ \text{(see p. 187.)}$$

This determines the value of the principal moment, viz.

$$\sqrt{\text{L}^2 + \text{M}^2 + \text{N}^2},$$

and also the values of a, b, c, which give the position of its axis.

If through any point (ξ, η, ζ) we draw three lines parallel to the axes of coordinates, the moments round these lines, which we may call L′, M′, N′, will evidently be found by putting $x - \xi$, $y - \eta$, $z - \zeta$ for x, y, z, in the values of L, M, N. This substitution gives

$$\text{L}' = \text{L} + \text{Z}\eta - \text{Y}\zeta,$$
$$\text{M}' = \text{M} + \text{X}\zeta - \text{Z}\xi,$$
$$\text{N}' = \text{N} + \text{Y}\xi - \text{X}\eta.$$

The principal moment with reference to this point will be

$$\sqrt{\text{L}'^2 + \text{M}'^2 + \text{N}'^2},$$

and if we investigate the conditions which make this a *minimum*,

we find (equating to 0 the partial differential coefficients with regard to ξ, η, ζ) equations which may be written as follows:

$$(9) \quad \frac{L + Z\eta - Y\zeta}{X} = \frac{M + X\zeta - Z\xi}{Y} = \frac{N + Y\xi - X\eta}{Z};$$

that is to say,

$$(10) \quad \frac{L'}{X} = \frac{M'}{Y} = \frac{N'}{Z} = \frac{\sqrt{L'^2 + M'^2 + N'^2}}{\sqrt{X^2 + Y^2 + Z^2}} = \frac{L'X + M'Y + N'Z}{X^2 + Y^2 + Z^2},$$

and therefore, (observing that $L'X + M'Y + N'Z = LX + MY + NZ$,)

$$\sqrt{L'^2 + M'^2 + N'^2} = \frac{LX + MY + NZ}{\sqrt{X^2 + Y^2 + Z^2}},$$

which determines the least principal moment.

If we equate two and two the expressions (9), we obtain by an easy transformation, (putting $X^2 + Y^2 + Z^2 = R^2$),

$$R^2\xi + NY - MZ = X(X\xi + Y\eta + Z\zeta),$$

and similar expressions for η, ζ; whence

$$\frac{R^2\xi + NY - MZ}{X} = \frac{R^2\eta + LZ - NX}{Y} = \frac{R^2\zeta + MX - LY}{Z},$$

which are the equations to the locus of the point ξ, η, ζ, that is, of the centres of least principal moments.

If there is a single resultant, let (ξ, η, ζ) be any point in its direction, and we must have

$$Z\eta - Y\zeta = L,$$
$$X\zeta - Z\xi = M,$$
$$Y\xi - X\eta = N.$$

If we multiply these equations by X, Y, Z, respectively, and add, we obtain the condition for a single resultant, viz.

$$(11) \qquad LX + MY + NZ = 0;$$

but if we subtract them, two and two, we find

$$MY - NZ = XY\zeta + XZ\eta - 2YZ\xi,$$

or $\qquad 3YZ\xi + MY - NZ = YZ\xi + XZ\eta + XY\zeta$:

similarly, $3XZ\eta + NZ - LX = YZ\xi + XZ\eta + XY\zeta$,

$$3XY\zeta + LX - MY = YZ\xi + XZ\eta + XY\zeta.$$

Hence, equating the first three members of these equations, and dividing by XYZ,

$$(12) \quad \frac{\xi - \frac{1}{3}\left(\frac{N}{Y} - \frac{M}{Z}\right)}{X} = \frac{\eta - \frac{1}{3}\left(\frac{L}{Z} - \frac{N}{X}\right)}{Y} = \frac{\zeta - \frac{1}{3}\left(\frac{M}{X} - \frac{L}{Y}\right)}{Z},$$

which are the equations to the resultant. If we write them for shortness,

$$(13) \quad \frac{\xi - a}{X} = \frac{\eta - \beta}{Y} = \frac{\zeta - \gamma}{Z},$$

it is easily seen that the length of the perpendicular upon it from the origin is

$$\sqrt{(\beta Z - \gamma Y)^2 + (\gamma X - \alpha Z)^2 + (\alpha Y - \beta X)^2} \cdot \frac{1}{\sqrt{X^2 + Y^2 + Z^2}}.$$

If we substitute for α, β, γ in this expression, it may be immediately reduced by the help of equation (11) to the following, viz.

$$\frac{\sqrt{L^2 + M^2 + N^2}}{\sqrt{X^2 + Y^2 + Z^2}},$$

as we might have anticipated *à priori*.

Jan. 2, 1839.   M. N. N.

[NOTE.—In a former Paper (see p. 189, equation 11, &c.) it was stated, that the envelope of the surface under consideration consisted, in certain cases, of three distinct surfaces. But it will be evident on a little reflection, that it really consists only of the points which are common to all the three, as in the case of the ellipsoids there mentioned. I may also observe, that in equation (1) of the same paper, h may be any function of x, y, z not containing a, b, c.]

VI.—ON THE IMPOSSIBLE LOGARITHMS OF QUANTITIES.

(*By* D. F. GREGORY, B.A. *Trinity College.*)

IN a Paper printed in the fourteenth volume of the *Transactions of the Royal Society of Edinburgh*, I gave a short sketch of what I conceive to be the true nature of Algebra, considered in its greatest generality; that it is the science of symbols, defined not by their nature, but by the laws of combination to which they are subject. In that paper I limited myself to a statement of the general view, without pretending to follow out all the conclusions to which such views would lead us: such an undertaking would be too extended for the limits of a memoir, and would involve a complete treatise on Algebra. It will not, however, be attempting too much to trace out, in one or two cases, some of the more important elucidations which this theory affords of several disputed and obscure points in Algebra, and therefore in the following pages I shall endeavour to point out the deductions which may be derived from the definition of the operation +, given in the paper above alluded to. I there stated, that we must not consider it merely as an affection of other symbols, which we call symbols of quantity, but as a distinct operation possessing certain properties peculiar to itself, and subject, like the more ordinary symbols, to be acted on by any other operations, such as the raising to powers,

&c. The definition of the operation represented by this symbol is, that

$$+ \; + \; = \; +,$$

which leads to the equation

$$(+)^r = +,$$

r being any integer. And this peculiarity—that the operation repeated any number of times gives the same result as when only performed once—is the origin of certain analytical anomalies, which do not at first sight appear to be connected.

The first of these is the fact of the existence of a plurality of roots of a quantity, when the corresponding powers have only one value. It seems a fair question, to ask the cause of so great a difference between two operations so analogous in their nature, but it is one which I have not seen anywhere discussed. The distinction is, I conceive, to be traced to the nature of the operation $+$, according to the definition of it which I have given; and much of the obscurity connected with the subject is due to an oversight, by which the existence of this $+$ is wholly overlooked. For it is not a, but $+ a$, which has a plurality of roots : and though these quantities are usually reckoned to be the same, this idea is founded on an illegitimate extension of a supposed relation in the science of number. I say *supposed*, because I hold, that even in Arithmetic a and $+ a$ are different, and ought not to be confounded—the former being an absolute operation, the other always a relative one, and consequently incapable of existing by itself. But however this may be, there is no doubt that it is entirely illegitimate to suppose that in all cases a and $+ a$ are the same, since generally we know not even what their meanings may be. Indeed, in Geometry the distinction is pretty broadly marked, since a represents a line considered with reference to magnitude only, $+ a$ with reference both to magnitude and direction. I therefore maintain, that in general symbolical Algebra we must never consider these quantities as identical; and if at any time we conceive the existence of the $+$, we must take cognizance of its existence throughout all our processes, subjecting it to the operations we may perform on the compound quantity. Now, that in the usual theory of the plurality of roots the existence of $+$ is supposed, though not always expressed, is easily shown from the very first case of plurality of values which occurs. It is argued that, since $a \times a = a^2$ and $- a \times - a = + a^2$ also, we have two values, a and $-a$, for $(a^2)^{\frac{1}{2}}$. But this, it will be seen, depends on the supposition that $+ a^2 = a^2$, since in the case of the product $- a \times - a$ the $+$ is exhibited. If, instead of saying $a \times a = a^2$, we were to say that $+ a \times + a = + a^2$, we should have undoubtedly $+ a^2$ as the result in both cases, and we are therefore entitled to say that $(+ a^2)^{\frac{1}{2}}$ has two values, $+ a$ and $- a$. The reason for this plurality is now very plain, for

$$(+ a^2)^{\frac{1}{2}} = + ^{\frac{1}{2}} (a^2)^{\frac{1}{2}} = + ^{\frac{1}{2}} a.$$

But from the definition of $+$ it appears that $+^{\frac{1}{2}}$ will be different according as we suppose the $+$ to be equivalent to the operation repeated an even or an odd number of times. In the former case it will be equal to $+$, in the latter to $-$. And generally, if we raise $+\ a$ to any power m, whether whole or fractional, we have

$$(+\ a)^m = +^m a^m.$$

Now, as from the definition of $+$ it appears that $+^r = +$, r being any integer, it is indeterminate which power of $+$ it may represent in any case, and therefore we must substitute $+^r$ for $+$, and then, assigning all integer values to r, discover how many values $+^{rm} a^m$ will acquire. So long as m is an integer, rm is an integer, and

$+^{rm} a^m$ has only one value; but if m be a fraction of the form $\dfrac{p}{q}$,

$+^{r\frac{p}{q}}$ will acquire different values, according as we assign different values to r. It will not, however, acquire an infinite number of values, since after r receives the value q, the values will recur in the same order. Hence the number of values of a quantity raised to a fractional power, is equal to the number of digits in the denominator of the index. It is to be observed, that we must never make $r = 0$, since that assumption is equivalent to supposing that the operation $+$ is not performed at all, which is contrary to our original supposition. From this we see, that the reason why there are a plurality of values for the roots of a quantity, is to be found in the nature of the operation $+$; and that it is only the compound operation $+\ a$, which admits of this plurality, a itself having only one value for each root. This view serves to explain an apparent difficulty which is noticed by various writers on Algebra. Since by the rule of signs $- \times -$ gives $+$, we ought to have

$$\sqrt{-\ a} \times \sqrt{-\ a} = \sqrt{+\ a^2} = \pm\ a;$$

whereas we know that it must be only $-\ a$.

Now this fallacy arises from the sign of the root not being made to affect the $+$ as well as the a. The process is really this,

$$\sqrt{-\ a} \times \sqrt{-\ a} = \sqrt{+\ a^2} = \sqrt{+}\ \sqrt{a^2} = -\ a;$$

for in this case we know how the $+$ has been derived, namely, from the product $- - = +$ or $-^2 = +$, which of course gives us $+^{\frac{1}{2}} = -$, there being here nothing indeterminate about the $+$.

It was in consequence of sometimes tacitly assuming the existence of $+$, and at another time neglecting it, that the errors in various trigonometrical expressions arose; and it was by the introduction of the factor $\cos 2r\pi + -^{\frac{1}{2}} \sin 2r\pi$ (which is equivalent to $+^r$) that Poinsot established the formulæ in a more correct and general shape. Thus the theorem of Demoivre that

$$(\cos \theta + -^{\frac{1}{2}} \sin \theta)^m = \cos m\theta + -^{\frac{1}{2}} \sin m\theta$$

should be written

$$\{+^{r}(\cos\theta + -\tfrac{1}{2}\sin\theta)\}^{m}\ \{= +^{rm}(\cos\theta + -\tfrac{1}{2}\sin\theta)^{m}$$
$$= (\cos 2r\pi + -\tfrac{1}{2}\sin 2r\pi)^{m}(\cos\theta + -\tfrac{1}{2}\sin\theta)^{m}$$
$$= \{\cos(2r\pi+\theta) + -\tfrac{1}{2}\sin(2r\pi+\theta)\}^{m}$$
$$= \cos m(2r\pi+\theta) + -\tfrac{1}{2}\sin m(2r\pi+\theta).$$

It will be seen from what I have said that I suppose the symbol + to play the same part which Professor Peacock ascribes to the symbol 1, when he says that it is the recipient of the affections of a^{m}; and that what that author considers to be the roots of unity I conceive to be the roots of +.

So far as the correctness of the formulæ is concerned, it makes but little difference which view is taken, if attention be always paid to the existence of this quantity on which the plurality of values depends, whether we denote it by the symbol 1 or +. But in the general Theory of Algebra there is a considerable difference; for 1 being an arithmetical symbol necessarily recals arithmetical notions; and as the circumstances in which its peculiar nature is evolved occur in general symbolical Algebra, and may be wholly independent of arithmetic, it is of importance to avoid the confusion which must be caused by the introduction into general symbolical Algebra of symbols limited in their signification.

The other point which I propose to elucidate at present, and which is the chief object of this paper, is the plurality of logarithms of quantities, which, although at first sight unconnected with what we have been discussing, will be found to depend also on the existence of a +, which is generally overlooked. This is closely connected also with the discussion concerning the logarithms of negative quantities, which attracted so much attention in the time of Euler, D'Alembert, and John Bernouilli, and the interest of which has been revived of late years by the researches of Vincent, Ohm, and Graves. Euler had apparently set the question at rest by demonstrating the existence of an infinite number of logarithms of a quantity, one only of which is possible; and the formula he gave was that

$$\log a = \mathrm{L}(a) + 2r\pi\sqrt{-1},$$

representing by $\mathrm{L}(a)$ the arithmetical logarithm of a.

Mr. Graves, by a different and very circuitous process, arrives at the result

$$\log(a) = \frac{\mathrm{L}(a) + 2r\pi\sqrt{-1}}{1 + 2r'\pi\sqrt{-1}},$$

the logarithms being taken with respect to the base ε for simplicity.

The correctness of this result is doubted by Professors Peacock and De Morgan, but it is corroborated by the researches of Sir W. Hamilton and Mr. Warren, as well as of M. Ohm. It is therefore both an interesting and an important question to determine which is the correct result, or at least to point out the cause of the differ-

ences between them. This I think the system I am advocating is able to do. But it is necessary first to lay down distinctly what is the meaning of the operation denoted by *log;* and this, according to my system, is done by defining its laws of combination. These are

$$\log x + \log y = \log (xy)\ldots\ldots(1)$$

$$\log (x^y) = y \log x \cdot \ldots\ldots(2),$$

where x and y are distributive and commutative operations,

$$\log a = 1\ldots\ldots(3),$$

which assumes the species to be that in which the base is a.

The first and third of these laws are the same as those given by Mr. Graves at the suggestion of Sir William Hamilton, but the second he has omitted; I know not whether from oversight, or from considering it to be unnecessary. I have retained it as I conceive it essential for a strict definition of the operation.

This being premised, I proceed to state the position which I lay down, and the truth of which I hope to be able to establish. It is, that the impossible parts of the general logarithms, whether of those given by Euler or by Mr. Graves, are the logarithms of the symbol $+$ which generally is overlooked in the expressions we use; and that the cause of the difference between the two formulæ for logarithms is, that in that of Euler *one* latent $+$ only, and in that of Mr. Graves *two* are exhibited.

This I think is almost apparent from Euler's own process, if we attend to the meaning of the symbols he employs. He substitutes for the number y the expression $(\cos 2r\pi + \sqrt{-1} \sin 2r\pi) y$, r being any integer which he considers to be equivalent to it; and then taking the logarithms with respect to ε, he says that

$$\log y = \mathrm{L}(y) + \log (\cos 2r\pi + \sqrt{-1} \sin 2r\pi),$$

where $\mathrm{L}(y)$ represents the arithmetical logarithm of y: and as

$$\cos 2r\pi + \sqrt{-1} \sin 2r\pi = \varepsilon^{2r\pi\sqrt{-1}},$$

we have $\log y = \mathrm{L}(y) + 2r\pi \sqrt{-1}.$

As r may receive any integer value, this expression has an infinite number of values, one only of which is possible in the case when $r = 0$. It will be seen that the correctness of this result depends essentially on the assumption that y and $(\cos 2r\pi + \sqrt{-1} \sin 2r\pi) y$ are identical: an assumption which at first it seems very natural to make, since the expression $\cos 2r\pi + \sqrt{-1} \sin 2r\pi$ is usually considered to be equal to unity. But if we suppose the quantities with which we are dealing to be general quantities, and not numbers merely, a numerical value of $\cos 2r\pi + \sqrt{-1} \sin 2r\pi$ can have no place in our investigation, and we must seek for its general algebraical meaning. Now in the paper previously referred to I have shewn that $+$ and $\cos 2\pi + \sqrt{-1} \sin 2\pi$ are algebraically equivalent, so that Euler's expression is equivalent to $+^r y$; and, as I

remarked before, we cannot assume y and $+y$ to be identical, so that Euler's assumption is not correct. If we do not suppose the existence of $+$ we have only one value for the logarithm of y; if we do suppose its existence, since it is indeterminate what power of $+$ it stands for, we must take all the possible cases, which is easily done by assigning to r all integer values from 0 to ∞. Thus

$$\log (+ y) = \log (+^r y) = \log (+^r) + \log y,$$

and as

$$+^r = (\cos 2\pi + \sqrt{-1} \sin 2\pi)^r = \cos 2r\pi + \sqrt{-1} \sin 2r\pi$$

$$\log (+ y) = 2r\pi \sqrt{-1} + \log y.$$

It must be observed that, as in the case of the powers of $+ y$, we must never suppose $r = 0$, since that is the same as supposing y not acted on by $+$, which is contrary to our original supposition.

Let us now consider Mr. Graves's method, stating as he does from the equation

$$y = a^x,$$

where a is the base of the system. If we assume y to stand for $+^r y$, we arrive at the same result as that of Euler. But we may also conceive a to stand for $+^{r'} a$, which is really, though not apparently, what is done by Mr. Graves, and then we obtain a very different result. The equation in this case becomes

$$+^r y = (+^{r'} a)^x,$$

and taking the logarithms with respect to ε for simplicity on both sides, we find

$$\log (+^r) + \log y = x \{ \log (+^{r'}) + \log a \}.$$

This gives

$$x = \frac{\log y + \log (+^r)}{\log a + \log (+^{r'})},$$

or, putting for $\log (+^r)$ and $\log (+^{r'})$ their values,

$$x = \frac{\log y + 2r\pi \sqrt{-1}}{\log a + 2r'\pi \sqrt{-1}},$$

which is Mr. Graves's result. We see that the difference between the methods of Euler and Mr. Graves consists in the nature of the base they assume. It may be remarked however that Euler seems to have had some idea of the view taken by Mr. Graves, as may be seen in his discussion of the Logarithmic Curve, vol. ii. p. 290 of the Latin edition, where he has anticipated the observations of M. Vincent, which nearly coincide in principle with those of Mr. Graves.

Mr. Peacock objects to the system adopted by Mr. Graves, because it involves a circulating function as base ; and I am inclined to agree with him. Since the base of the system is now $+^{r'} a$ instead of a; the supposition of a change in the value of r'

corresponds to a change in the base, and therefore in the whole system of logarithms, so that the series of values of x corresponding to the different values of r' have as little connexion with each other as if they belonged to systems whose bases were b, c, d, or any other quantities. This of course depends on our believing that it is $(+a)^n$ and not a^n which has a plurality of values, and this I think I have satisfactorily shown. I may observe that if we are to allow a variable base as $+r'a$, we might as well use such quantities as $\sin^{-1}a$, $\cos^{-1}a$ as bases, and reckon the logarithms corresponding to different values to belong to the same system; but this is what I believe no one would admit. The defect of not considering the existence of $+$ will perhaps appear more clearly if we analyse the reasoning by which both M. Vincent and Mr. Graves think that they establish that in certain cases there is a common logarithm for positive and negative numbers. They argue that since $\varepsilon^{\frac{1}{2}}$ or $\sqrt{e}$ has two values which we may call $+n$ and $-n$, we have therefore

$$+n = \varepsilon^{\frac{1}{2}}, \quad -n = \varepsilon^{\frac{1}{2}},$$

and, from the ordinary definition of logarithms, $\frac{1}{2}$ must be the logarithm both of $+n$ and $-n$. So indeed, it is, but only when referred to different systems: for, as I maintain, $+n$ and $-n$ are not both values of $\varepsilon^{\frac{1}{2}}$, but one is the value of $(+^2\varepsilon)^{\frac{1}{2}}$ and the other of $(+\varepsilon)^{\frac{1}{2}}$, so that $\frac{1}{2}$ is the logarithm of $+n$ in the system whose base is $+^2\varepsilon$, and of $-n$ in the system whose base is $+\varepsilon$. The same reasoning may be generally extended to such cases as $(+a)^{\frac{1}{n}}$ which admits of n values, and consequently of n quantities, which have a common logarithm $\dfrac{1}{n}$, but in each case referred to a different base. When n is even, one value will be positive and the other negative, all the others being impossible; and the positive and negative values are the only two of which M. Vincent takes notice when discussing the question. It might, perhaps, have weakened his belief in the correctness of the results, if he had come to the conclusion, as he ought to have done, that the same logarithm corresponded to positive, negative, and impossible quantities. These last he seems quite to have overlooked, which may have arisen from his having adopted, with many other mathematicians, the name of *imaginary* quantities. I adhere to the name *impossible* instead of *imaginary*, because the latter involves an idea which I conceive to be very deleterious in analysis. We may be unable to perform an operation though it be by no means an *imaginary* one; and indeed all that we can say of those quantities which have this name affixed to them is, that they are *uninterpretable in arithmetic*. For this reason, if I were permitted to propose a change, I should prefer to call these quantities "operations uninterpretable in arithmetic;" as this involves no theory of their nature, but only expresses what is a fact.

That, according to the system which I adopt, there cannot be a logarithm common to both positive and negative quantities, is easily shown. A positive quantity may be generally expressed by

$$+^r a:$$

the logarithm of which is

$$\log a + \log (+^r) = \log a + 2r\pi \sqrt{-1}.$$

A negative quantity may be expressed by

$$+^{\frac{2r+1}{2}} a:$$

the logarithm of which is

$$\log a + \log (+^{\frac{2r+1}{2}}) = \log a + \frac{2r+1}{2} \pi \sqrt{-1}.$$

And these two expressions can never coincide; nor can either ever lose its impossible part, since we are not at liberty to make $r = 0$ in the first case, or $= -\frac{1}{2}$ in the second.

It is somewhat remarkable, that Mr. Peacock has been led into the same error as M. Vincent and Mr. Graves, respecting the coincidence in some cases of the logarithms of positive and negative quantities. As the cause of his error has reference to the remark which I have just made, and is not very easy to be detected, I shall point it out more particularly.

He considers $-a^m$ to be equivalent to $-1(+a)^m$, which gives

$$\log - (a)^m = \log (-1) + \log (+a)^m$$
$$= (2r + 2mr' + 1)\pi \sqrt{-1} + m \log a.$$

He then supposes $m = \frac{p}{2n}$ where p is prime to n, $r' = -n$, and $r = \frac{p-1}{2}$; and as these values make the multiplier of $\pi \sqrt{-1}$ vanish, he concludes that the logarithm of $-(a)^m$ coincides with that of a^m, since it becomes $m \log a$. Now on this it is to be observed, that since m affects the $+$ in $(+a)^m$, $-a^m$ is really equal to $-1 . +^m . a^m$, or, putting the general values for $-$ and $+$, to

$$+^{\frac{2r+1}{2}} (+^{r'})^m a^m.$$

In this expression, if we make $m = \frac{p}{2n}$, $r' = -n$, $r = \frac{p-1}{2}$, it becomes

$$+^{\frac{p}{2}} . +^{-\frac{p}{2}} . a^{\frac{p}{2n}};$$

and as $+^{\frac{p}{2}}$ and $+^{-\frac{p}{2}}$ are inverse operations, they destroy each other, and we have simply $a^{\frac{p}{2n}}$; the logarithm of which is, as it should be, possible. But these assumptions as to the values of m, r, and r', are plainly not allowable, since they imply, as we have

seen, that a^m is not affected by $-$ at all, which is contrary to the original supposition. Hence we perceive that Mr. Peacock's argument for the existence of logarithms common to positive and negative quantities, being based on an unlawful assumption, falls to the ground.

If it be allowable to assume any quantity as base for a system of logarithms, we might, instead of $+^r a$ when r is an integer, take the same quantity, supposing r to be a fraction. We should then have possible quantities corresponding to impossible logarithms, and impossible quantities to possible logarithms; but the subject does not appear to be of sufficient interest to require an extended discussion.

In conclusion, I will recapitulate the conclusions to which I have been led by this mode of considering the symbol $+$.

1. A simple distributive and commutative operation has only one root, but if it be compounded with $+$ it has a plurality of roots depending on the indeterminate nature of $+$.

2. If the base of a system of logarithms and the number be simple distributive and commutative operations, there is only one corresponding logarithm; but if the number of the form be $+^r y$, there is an infinite number of logarithms.

3. If the base of the system be of the form $+^r a$, we are only allowed to assign one value to r, (as otherwise we alter the system,) and then there will be no plurality of logarithms.

4. The logarithms of $+ a$ and $- a$ are in all cases different, and neither ever coincide with that of a.

5. The impossible parts of the logarithms, as usually given, are the logarithms of $+$ and of $-$.

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VIII.—ON THE EQUATION WHICH DETERMINES THE STABILITY OF THE PLANETARY EXCENTRICITIES.

THE equation which enables us to prove the stability of the excentricities of the planetary orbits, may be deduced as follows.

We have

$$\frac{de_1}{dt} = - \frac{n_1 a_1}{\mu e_1} (1 - e_1^2) \frac{dR_1}{d\varepsilon_1} + \frac{n_1 a_1}{\mu e_1} \sqrt{1 - e_1^2} \left(\frac{dR_1}{d\varepsilon_1} + \frac{dR_1}{d\omega_1} \right)$$

$$= \frac{n_1 a_1}{\mu e_1} (1 - e_1^2) \frac{dR_1}{d\varepsilon_1} + \frac{n_1 a_1}{\mu e_4} \sqrt{1 - e_1^2} \frac{dR_1}{d\omega_1}.$$

Now, wherever ε_1 occurs in R_1, $n_1 t + \varepsilon_1$ occurs, that is, ε_1 occurs only in periodical terms; hence, neglecting periodical terms,

$$\frac{m}{n_1 a_1} e_1 \frac{de_1}{dt} = \frac{m}{\mu} \sqrt{1 - e_1^{\,2}} \frac{dR_1}{d\theta_1};$$

or, neglecting terms of a higher order than the second, and observing that $\dfrac{m}{n_1 a_1} e_1 \dfrac{de_1}{dt}$ is itself of the first order, we have

$$\frac{m}{n_1 a_1} e_1 \frac{de_1}{dt} = \frac{m}{\mu} \frac{dR}{d\theta}.$$

Now,

$$R = - \frac{m'}{\sqrt{r^2 - 2rr' \cos(\theta - \theta') + r'^2}} + \frac{m'r \cos(\theta - \theta')}{r'^2};$$

$$\therefore \frac{dR}{d\theta} = - \frac{m' \sin(\theta - \theta') rr'}{\{r^2 - 2rr' \cos(\theta - \theta') + r'^2\}^{\frac{3}{2}}} - \frac{m'r \sin(\theta - \theta')}{r'^2};$$

$$\therefore \frac{m}{n_1 a_1} e_1 \frac{de_1}{dt} = k \sin(\theta - \theta') - \frac{m\,m'}{\mu} \frac{r \sin(\theta - \theta')}{r'^2},$$

where k is a quantity the same for both m and m'.

Similarly,

$$\frac{m'}{n_1' a_1'} e_1' \frac{de_1'}{dt} = - k \sin(\theta - \theta') + \frac{m\,m'}{\mu} \frac{r' \sin(\theta - \theta')}{r'^2};$$

$$\therefore \frac{m}{n_1 a_1} e_1 \frac{de_1}{dt} + \frac{m'}{n_1' a_1'} e_1' \frac{de_1'}{dt} = - \frac{m\,m'}{\mu} \left(\frac{r}{r'^2} - \frac{r'}{r^2} \right) \sin(\theta - \theta'),$$

we have

$$\sin(\theta - \theta') = \sin\{nt + \varepsilon - (n't + \varepsilon') + z\}$$
$$= \sin\{nt + \varepsilon - (n't + \varepsilon')\} \cos z + \cos\{nt + \varepsilon - (n't + \varepsilon')\} \sin z,$$

where $\genfrac{}{}{0pt}{}{\sin}{\cos} z$ consist each of a series of terms respectively of the form

$$P \genfrac{}{}{0pt}{}{\sin}{\cos} \{p(nt + \varepsilon - \omega) - q(n't + \varepsilon' - \omega')\},$$

where P is a quantity involving $e^p e'^q$; and $\dfrac{r}{r'^2}$, $\dfrac{r'}{r^2}$ each consists of a series of terms of the form

$$P \cos \{p(nt + \varepsilon - \omega) - q(n't + \varepsilon' - \omega')\},$$

where P involves $e^p e'^q$, so that $\cos z \cdot \dfrac{r}{r'^2}$ and $\cos z \cdot \dfrac{r'}{r^2}$ will be made up of terms of the form

$$P \cos \{p(nt + \varepsilon - \omega) - q(n't + \varepsilon' - \omega')\};$$

and $\sin z \cdot \dfrac{r}{r'^2}$, as also $\sin z \cdot \dfrac{r'}{r^2}$, will be made up of terms of the form

$$\mathrm{P} \sin \{p\,(nt + \varepsilon - \omega) - q\,(n't + \varepsilon' - \omega')\},$$

P in each case involving $e^p e'^q$. Now the only way in which a constant term can arise in $\sin (\theta - \theta')\,\dfrac{r}{r'^2}$, is by combination of

$$\begin{matrix}\sin\\\cos\end{matrix} \{nt + \varepsilon - (n't + \varepsilon')\} \text{ with } \begin{matrix}\sin\\\cos\end{matrix} \{nt + \varepsilon - \omega - (n't + \varepsilon' - \omega')\},$$

so that $\sin (\theta - \theta')\,\dfrac{r}{r'^2}$ can only contain one such term, which will be of the form

$$ee' \{\mathrm{A} + \phi\,(e, e')\} \cos (\omega - \omega')\},$$

where ϕ denotes an integral function. Similarly, $\sin (\theta - \theta')\dfrac{r'}{r^2}$ can only contain one constant term, which will be

$$e\,e' \{\mathrm{A} + \psi\,(e, e')\} \cos (\omega - \omega');$$

hence, rejecting quantities of a higher order than the second, there is no constant term in $\sin (\theta - \theta') \left(\dfrac{r}{r'^2} - \dfrac{r'}{r^2} \right)$;

$$\therefore \frac{m}{na}\, e_1 \frac{de_1}{dt} + \frac{m'}{n'a'}\, e_1' \frac{de_1'}{dt} = 0;$$

$$\therefore \frac{m}{na}\, e_1{}^2 + \frac{m'}{n'a'}\, e_1'^2 = \mathrm{C}.$$

$\lambda.$

IX.—GENERAL FORMULÆ FOR THE CHANGE OF THE INDEPENDENT VARIABLE.

GIVEN an expression involving the differential coefficients of u with respect to x, it is required to change it into an equivalent expression involving the differential coefficients of u with respect to y, y being a given function of x.

Suppose $u = \mathrm{F}\,(y)$; then, a being an arbitrary quantity,

$$\mathrm{F}\,(y) = \mathrm{F}\,\{a + (y - a)\}$$

$$= \mathrm{F}\,(a) + \frac{y - a}{1}\, \mathrm{F}_1(a) + \frac{(y-a)^2}{1 \cdot 2}\, \mathrm{F}_2(a) + \frac{(y-a)^3}{1 \cdot 2 \cdot 3}\, \mathrm{F}_3(a) + \dots$$

$$\therefore \frac{d^n \mathrm{F}\,(y)}{dx^n} = \frac{1}{1} \frac{d^n\,(y-a)}{dx^n}\, \mathrm{F}_1\,(a) + \frac{1}{1 \cdot 2} \frac{d^n\,(y-a)^2}{dx^n}\, \mathrm{F}_2\,(a)$$

$$+ \frac{1}{1 \cdot 2 \cdot 3} \frac{d^n\,(y-a)^3}{dx^n}\, \mathrm{F}_3\,(a) + \dots$$

Suppose the differentiation performed, and then put $a = y$;

$$\therefore \frac{d^n u}{dx^n} = \frac{1}{1} \frac{du}{dy} \frac{d^n (y-a)}{dx^n} + \frac{1}{1 \cdot 2} \frac{d^2 u}{dy^2} \cdot \frac{d^n (y-a)^2}{dx^n}$$

$$+ \frac{1}{1 \cdot 2 \cdot 3} \frac{d^3 u}{dy^3} \cdot \frac{d^n (y-a)^3}{dx^n} + \ldots$$

where y is to be substituted for a after differentiation. The number of terms will be finite, because the nth differential coefficients of the powers of $y - a$ higher than the nth will vanish when y is put for a.

It remains to express $\dfrac{d^n (y-a)^p}{dx^n}$ in terms of the differential co-efficient of y with regard to x.

Let y' be the value of y when x is changed to $x + h$. Then, by Taylor's theorem,

$$y' - a = y - a + \frac{h}{1} \frac{dy}{dx} + \frac{h^2}{1 \cdot 2} \frac{d^2 y}{dx^2} + \frac{h^3}{1 \cdot 2 \cdot 3} \frac{d^3 y}{dx^3} + \ldots$$

also

$$(y'-a) = (y-a) + \frac{h}{1} \frac{d(y-a)^p}{dx} + \frac{h^2}{1 \cdot 2} \frac{d^2(y-a)^p}{dx^2} + \frac{h^3}{1 \cdot 2 \cdot 3} \frac{d^3(y-a)^p}{dx^3} + \ldots$$

Hence $\dfrac{d^n (y-a)^p}{dx^n} = 1 \cdot 2 \ldots n \times$ coefficient of h^n in

$$\left(y - a + \frac{n}{1} \frac{dy}{dx} + \frac{h^2}{1 \cdot 2} \frac{d^2 y}{dx^2} + \frac{h^3}{1 \cdot 2 \cdot 3} \frac{d^3 y}{dx^3} + \ldots \right)^p.$$

Now, by the polynomial theorem, the coefficient of h^n in the expansion of this is

$$\frac{1 \cdot 2 \cdot 3 \ldots p}{(1 \cdot 2 \ldots \alpha)(1 \cdot 2 \ldots \beta)(1 \cdot 2 \ldots \gamma) \ldots}$$

$$\times \left(\frac{1}{1 \cdot 2 \ldots \lambda} \frac{d^\lambda y}{dx^\lambda} \right)^\alpha \left(\frac{1}{1 \cdot 2 \ldots \mu} \frac{d^\mu y}{dx^\mu} \right)^\beta \left(\frac{1}{1 \cdot 2 \ldots \nu} \frac{d^\nu y}{dx^\nu} \right)^\gamma \ldots$$

where $\alpha + \beta + \gamma + \ldots = p$;

and $\alpha\lambda + \beta\mu + \gamma\nu + \ldots = n$,

therefore the general term of $\dfrac{d^n (y-a)^p}{dx^n}$ is

$$\frac{1 \cdot 2 \cdot 3 \ldots p \times 1 \cdot 2 \cdot 3 \ldots n}{1.2 \ldots \alpha \times 1.2 \ldots \beta \times 1.2 \ldots \gamma \times \ldots \times (1.2 \ldots \lambda)^\alpha (1.2 \ldots \mu)^\beta (1.2 \ldots \nu)^\gamma \ldots}$$

$$\times \left(\frac{d^\lambda y}{dx_\lambda} \right)^\alpha \left(\frac{d^\mu y}{dx_\mu} \right)^\beta \left(\frac{d^\nu y}{dx_\nu} \right)^\gamma \ldots$$

the quantities α, β, γ, &c. and λ, μ, ν, &c. being subject to the two conditions above. It must be observed, that for $\alpha = 0$, $1 \cdot 2 \ldots \alpha$ becomes 1, and for $\lambda = 0$, $1 \cdot 2 \ldots \lambda$ becomes 1, and $\dfrac{d^\lambda y}{dx^\lambda}$ becomes

$y - a$. Hence, the terms in which any of the quantities λ, μ, ν, &c. are 0, may be neglected, because they vanish when y is put for a.

$\int\!\!\int$.

X.—MATHEMATICAL NOTES.

1. THE following is an easy method of obtaining the general differences of $\sin x$ and $\cos x$.

$$\Delta \sin x = \sin(x + h) - \sin x = \sin x (\cos h - 1) + \cos x \sin h$$

$$= 2 \sin \frac{h}{2} \left(- \sin x \sin \frac{h}{2} + \cos x \cos \frac{h}{2} \right).$$

But $-\sin x = \cos\left(x + \frac{\pi}{2}\right)$. $\cos x = \sin\left(x + \frac{\pi}{2}\right)$;

therefore

$$\Delta \sin x = 2 \sin \frac{h}{2} \left\{ \cos\left(x + \frac{\pi}{2}\right) \sin \frac{h}{2} + \sin\left(x + \frac{\pi}{2}\right) \cos \frac{h}{2} \right\}$$

$$= 2 \sin \frac{h}{2} \sin \left(x + \frac{\pi + h}{2}\right) = \left(2 \sin \frac{h}{2}\right) D^{\frac{1}{2}(\pi + h)} \sin x;$$

therefore $\Delta^n \sin x = \left(2 \sin \frac{h}{2}\right)^n D^{\frac{n}{2}(\pi + h)} \sin x$

$$= \left(2 \sin \frac{h}{2}\right)^n \sin \left(x + \frac{n}{2}(\pi + h)\right).$$

Similarly,

$$\Delta^n \cos x = \left(2 \sin \frac{h}{2}\right)^n \cos \left(x + \frac{n}{2}(\pi + h)\right).$$

These results are true whether n be positive or negative, whole or fractional.

In differentials the same method may be employed,

$$\frac{d}{dx} \sin x = \cos x = \sin \left(x + \frac{\pi}{2}\right) = D^{\frac{\pi}{2}} \sin x,$$

and therefore

$$\left(\frac{d}{dx}\right)^n \sin x = D^{n\frac{\pi}{2}} \sin x = \sin \left(x + n\frac{\pi}{2}\right),$$

whatever n may be.

β.

2. It seems not to be generally known, that the equation

$$1 \cdot 2 \cdot 3 \ldots n = n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1 \cdot 2}(n-2)^n - \ldots$$

which is used in proving Sir John Wilson's theorem respecting prime numbers, can be deduced immediately from the theorems of common Algebra. The following is the method.

By the Binomial Theorem,

$$(\varepsilon^x - 1)^n = \varepsilon^{nx} - \frac{n}{1}\varepsilon^{(n-1)x} + \frac{n(n-1)}{1 \cdot 2}\varepsilon^{(n-2)x} - \ldots$$

Substitute for each exponential its expansion according to powers of x, and equate the coefficients of x^n on the two sides. That on the first side, or $\left(x + \frac{x^2}{1 \cdot 2} + \ldots\right)^n$, is evidently 1; the coefficient of x^n in ε^{nx} is $\frac{n^n}{1 \cdot 2 \cdot 3 \ldots n}$; in $\varepsilon^{(n-1)x}$, $\frac{(n-1)^n}{1 \cdot 2 \cdot 3 \ldots n}$; in $\varepsilon^{(n-2)x}$, $\frac{(n-2)^n}{1 \cdot 2 \cdot 3 \ldots n}$, &c. Hence,

$$1 = \frac{n^n}{1 \cdot 2 \cdot 3 \ldots n} - \frac{n}{1} \cdot \frac{(n-1)^n}{1 \cdot 2 \cdot 3 \ldots n} + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{(n-2)^n}{1 \cdot 2 \cdot 3 \ldots n} - \ldots$$

or $1 \cdot 2 \cdot 3 \ldots n = n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1 \cdot 2}(n-2)^n - \ldots$

In the same way it is seen that

$$n^m - \frac{n}{1}(n-1)^m + \frac{n(n-1)}{1 \cdot 2}(n-2)^m - \ldots$$

is zero if m and n be any integer, of which n is the greater.

$$\gamma.$$

3. The equation to the tangent of the ellipse given in Art. 2. of No. I., furnishes a ready solution of the problem, To find the locus of the intersections of pairs of tangents to an ellipse, which are always parallel to conjugate diameters.

The equation to one tangent being

$$y - ax = \sqrt{a^2a^2 + b^2};$$

that to the other is

$$y + \frac{b^2}{a^2a} = \sqrt{\frac{a^2b^4}{a^4a^2} + b^2},$$

since it is parallel to the conjugate diameter. Multiplying up, this becomes

$$a^2ay + b^2x = ab\sqrt{b^2 + a^2a^2};$$

and multiplying the first equation by ab, it becomes

$$aby - ab\,ax = ab\sqrt{a^2a^2 + b^2}.$$

Squaring these two equations, and adding, we get

$$a^2y^2 (a^2a^2 + b^2) + b^2x^2 (a^2a^2 + b^2) = 2a^2b^2 (a^2a^2 + b^2),$$
$$\text{or} \quad a^2y^2 + b^2x^2 = 2a^2b^2.$$

$$\delta.$$

4. *Decomposition of Rational Fractions.* If the denominator of a rational fraction contain equal roots, the equivalent fractions may be easily determined by a process similar to Maclaurin's theorem.

Let the fraction be $\dfrac{f(x)}{(x-a)^n}$; then

$$f(x) = f(x+z-a) \text{ (when } z=a) = f(z+x-a) \text{ (when } z=a),$$

or, expanding by Taylor's theorem,

$$f(x) = f(z) + (x-a)\frac{d}{dz}f(z) + \dots \frac{(x-a)^{n-1}}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}f(z) + \text{R}.$$

$$\text{(when } z = a).$$

If $f(x)$ be an integral function of x of a degree, at least one less than the degree of the denominator, $\text{R} = 0$, since all the terms after the nth vanish. If $f(x)$ be fractional, we must determine R in the usual way. Dividing now by $(x-a)^n$, we have

$$\frac{f(x)}{(x-a)^n} = \frac{f(z)}{(x-a)^n} + \frac{1}{(x-a)^{n-1}}\frac{d}{dz}f(z) + \dots$$
$$+ \frac{1}{(n-1)!(x-a)}\frac{d^{n-1}}{dz^{n-1}}f(z) + \frac{\text{R}}{(x-a)^n},$$
$$\text{(when } z = a).$$

$$\phi.$$

5. Note on Art. VII.—The view taken in this article of the cause of the plurality of values of a root of $+ a$, may perhaps be more clearly explained by stating, that the idea entertained is, that there is really no plurality of *roots* of one quantity, but that there is an indeterminateness as to the quantity, the root of which is taken: and the same is to be said of the logarithms.

$$\phi.$$

Fig 1.

Fig 2.

Fig 3.

Fig 6

Fig 5

Fig 4.

Fig. 9

Fig. 8

Fig. 7

Fig. 11

Fig. 12

Fig 10